

INNERNESS OF CONTINUOUS DERIVATIONS ON ALGEBRAS OF LOCALLY MEASURABLE OPERATORS

A. F. BER, V. I. CHILIN, AND F. A. SUKOCHEV

ABSTRACT. It is established that every derivation continuous with respect to the local measure topology acting on the $*$ -algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated with a von Neumann algebra \mathcal{M} is necessary inner. If \mathcal{M} is a properly infinite von Neumann algebra, then every derivation on $LS(\mathcal{M})$ is inner. In addition, it is proved that any derivation on \mathcal{M} with values in Banach \mathcal{M} -bimodule of locally measurable operators is inner.

1. INTRODUCTION

One of the important results of the theory of derivations in Banach bimodules is the Theorem of J. R. Ringrose on automatic continuity of every derivation from a C^* -algebra \mathcal{M} into a Banach \mathcal{M} -bimodule [25]. This theorem extends the well-known result that every derivation of a C^* -algebra \mathcal{M} is automatically norm continuous [27]. In the case when \mathcal{M} is a AW^* -algebra (in particular, W^* -algebra), every derivation on \mathcal{M} is inner [23], [27]. Numerous results on continuity of derivations in Banach algebras are given in [13].

Significant examples of W^* -modules are non-commutative rearrangement invariant spaces of measurable operators affiliated with a von Neumann algebra. At the present time the theory of rearrangement invariant spaces is actively developed [17], [21], and it gives useful applications both in the geometry of Banach spaces and in the theory of unbounded operators. Every non-commutative rearrangement invariant space is a solid linear space in the $*$ -algebra $S(\mathcal{M}, \tau)$ of all τ -measurable operators affiliated with a von Neumann algebra \mathcal{M} , where τ is a faithful normal semifinite trace on \mathcal{M} [24]. The algebra $S(\mathcal{M}, \tau)$ equipped with the natural topology t_τ of convergence in measure generated by the trace τ is a complete metrizable topological algebra. In its turn the algebra $S(\mathcal{M}, \tau)$ represents a solid $*$ -subalgebra of the $*$ -algebra $LS(\mathcal{M})$ of all locally measurable operators, affiliated with a von Neumann algebra \mathcal{M} [28], [31]. The $*$ -algebras $LS(\mathcal{M})$ and $S(\mathcal{M}, \tau)$

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as well as the algebra $S(\mathcal{M})$ of measurable operators affiliated with \mathcal{M} [29], are useful examples of EW^* -algebras of unbounded operators [15]. Moreover, in [12] it is established that every EW^* -algebra \mathcal{A} with the bounded part $\mathcal{A}_b = \mathcal{M}$ is a solid $*$ -subalgebra in the $*$ -algebra $LS(\mathcal{M})$, i.e. $LS(\mathcal{M})$ is the greatest EW^* -algebra of all EW^* -algebras with the bounded part coinciding with \mathcal{M} . In addition, the algebra $LS(\mathcal{M})$ with the natural topology $t(\mathcal{M})$ of convergence locally in measure is a complete topological $*$ -algebra [31].

Every EW^* -algebra $(\mathcal{A}, t(\mathcal{M}))$ is an example of GB^* -algebras, which properties are well investigated in [16]. The bounded part $\mathcal{A}(\mathcal{B}_0)$ of every GB^* -algebra \mathcal{A} is a C^* -algebra [16] and the algebra \mathcal{A} by itself is a topological bimodule over $\mathcal{A}(\mathcal{B}_0)$. In the case when \mathcal{A} is an EW^* -algebra the bounded part $\mathcal{A}(\mathcal{B}_0)$ coincides with \mathcal{A}_b . A natural development of J. R. Ringrose [25] and Sakai-Kadison Theorems [27] is the study of the properties of continuity and innerness of derivations acting from $\mathcal{A}(\mathcal{B}_0)$ into \mathcal{A} .

This problem is directly connected with researches on derivations on algebras of unbounded operators. One of the first work in this field became the paper by C. Brodel and G. Lassher [10], where it was established that every derivation on a complete O^* -algebra of unbounded operator is spatial. Similar results for other classes of locally convex algebras of unbounded operators are obtained in [18, 19].

Since every EW^* -algebra \mathcal{A} with the bounded part $\mathcal{A}_b = \mathcal{M}$ is a topological $*$ -algebra of unbounded operators with respect to the non locally convex topology $t(\mathcal{M})$, the problem of innerness and $t(\mathcal{M})$ -continuity of a derivation from \mathcal{M} into \mathcal{A} seems natural.

In [6] it is proven that each derivation $\delta : \mathcal{M} \rightarrow \mathcal{A}$ extends up to a derivation from $LS(\mathcal{M})$ into $LS(\mathcal{M})$. In this respect, we should describe properties of $t(\mathcal{M})$ -continuity and innerness of derivations $\delta : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$.

In the setting of commutative W^* -algebras this problem is fully resolved in [5]. In the setting of von Neumann algebras of type I , a thorough treatment of this problem may be found in [1] and [7]. The papers [1, 5] contain examples of non-inner derivations of the $*$ -algebra $LS(\mathcal{M})$, which are not continuous with respect to the topology $t(\mathcal{M})$ of convergence locally in measure on $LS(\mathcal{M})$. On the other hand, it is shown in [1] that in the special case when \mathcal{M} is a properly infinite von Neumann algebra of type I , every derivation of $LS(\mathcal{M})$ is continuous with respect to the local measure topology $t(\mathcal{M})$. Using a completely different technique, a similar result was also obtained in [7] under the additional assumption that the predual space \mathcal{M}_* to \mathcal{M} is separable. It is of interest to observe that an analogue of this result (that is the continuity of an arbitrary derivation of $(LS(\mathcal{M}), t(\mathcal{M}))$) also holds for any von Neumann algebra \mathcal{M} of type III [2]. In [2] the following problem is formulated (Problem 3): Let \mathcal{M} be a von Neumann algebra of

type II and let τ be a faithful normal semifinite trace on \mathcal{M} . Is any derivation on a $*$ -algebra $S(\mathcal{M}, \tau)$ equipped with the measure topology t_τ necessarily continuous? In [3] this problem is solved affirmatively for a properly infinite algebra \mathcal{M} . In view of the example we mentioned above, a natural problem (similar to Problem 3 from [2]) is whether any derivation of a $*$ -algebra $LS(\mathcal{M})$ is necessarily continuous with respect to the topology $t(\mathcal{M})$, where \mathcal{M} is a properly infinite von Neumann algebra of type II . In [6] it is given the positive solution of this problem. In fact, in [6] it is established a much stronger result that any derivation $\delta : \mathcal{A} \rightarrow LS(\mathcal{M})$, where \mathcal{A} is any EW^* -subalgebra in $LS(\mathcal{M})$ with $\mathcal{A}_b = \mathcal{M}$, is necessarily continuous with respect to the topology $t(\mathcal{M})$ in the case when \mathcal{M} is properly infinite von Neumann algebra.

In this respect the problem of innerness of $t(\mathcal{M})$ -continuous derivations $\delta : \mathcal{A} \rightarrow \mathcal{A}$ (EW^* -version of Sakai-Kadison Theorem) naturally arises. For a von Neumann algebra of type I and III this problem is solved in [1, 2]. In the present paper it is proven that for every $t(\mathcal{M})$ -continuous derivation δ acting on an EW^* -algebra \mathcal{A} with the bounded part $\mathcal{A}_b = \mathcal{M}$, there exists $a \in \mathcal{A}$, such that $\delta(x) = ax - xa = [a, x]$ for all $x \in \mathcal{A}$, i.e. derivation δ is inner. Moreover, it is established that every derivation on a von Neumann algebra \mathcal{M} with values in a noncommutative rearrangement invariant space $\mathcal{E} \subset S(\mathcal{M}, \tau)$ is necessary inner.

The proof proceeds in several stages. In section 3 we introduce the notion of λ -system for a self-adjoint derivation $\delta : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$ and study the properties of this λ -system, in particular, it is given the estimate on the value of dimensional function for the support of λ -system. After that in section 4 it is given the proof of the main result of the present paper (Theorem 4.1) on innerness of every $t(\mathcal{M})$ -continuous derivation $\delta : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$. In particular, in view of the result of [6], it is shown that for a properly infinite von Neumann algebra \mathcal{M} every derivation on $LS(\mathcal{M})$ is inner. In section 5 we give applications of Theorem 4.1, establishing the innerness of all $t(\mathcal{M})$ -continuous derivations acting on an EW^* -algebra \mathcal{A} with the bounded part $\mathcal{A}_b = \mathcal{M}$. In last section 6 we introduce the class of Banach \mathcal{M} -bimodules of locally measurable operators $\mathcal{E} \subset LS(\mathcal{M})$. This class contains all noncommutative rearrangement invariant spaces. It is proven that every derivation $\delta : \mathcal{M} \rightarrow \mathcal{E}$ is inner, i.e. it has a form $\delta(x) = [d, x] = \delta_d(x)$ for all $x \in \mathcal{M}$ and some $d \in \mathcal{E}$, in particular, δ is a continuous derivation from $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ into $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$. In addition, the operator $d \in \mathcal{E}$ such that $\delta = \delta_d$ may be chosen so that $\|d\|_{\mathcal{E}} \leq 2\|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$.

We use terminology and notations from the von Neumann algebra theory [20, 27, 30] and the theory of locally measurable operators from [22, 28, 31].

2. PRELIMINARIES

Let H be a Hilbert space, let $B(H)$ be the $*$ -algebra of all bounded linear operators on H , and let $\mathbf{1}$ be the identity operator on H . Given a von Neumann algebra \mathcal{M} acting on H , denote by $\mathcal{Z}(\mathcal{M})$ the centre of \mathcal{M} and by $\mathcal{P}(\mathcal{M}) = \{p \in \mathcal{M} : p = p^2 = p^*\}$ the lattice of all projections in \mathcal{M} . Let $\mathcal{P}_{fin}(\mathcal{M})$ be the set of all finite projections in \mathcal{M} .

A linear operator $x : \mathfrak{D}(x) \rightarrow H$, where the domain $\mathfrak{D}(x)$ of x is a linear subspace of H , is said to be *affiliated* with \mathcal{M} if $yx \subseteq xy$ for all y from the commutant \mathcal{M}' of algebra \mathcal{M} .

A densely-defined closed linear operator x (possibly unbounded) affiliated with \mathcal{M} is said to be *measurable* with respect to \mathcal{M} if there exists a sequence $\{p_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathfrak{D}(x)$ and $p_n^\perp = \mathbf{1} - p_n \in \mathcal{P}_{fin}(\mathcal{M})$ for every $n \in \mathbb{N}$, where \mathbb{N} is the set of all natural numbers. Let us denote by $S(\mathcal{M})$ the set of all measurable operators.

Let $x, y \in S(\mathcal{M})$. It is well known that $x + y$ and xy are densely-defined and preclosed operators. Moreover, the closures $\overline{x + y}$ (strong sum), \overline{xy} (strong product) and x^* are also measurable, and equipped with this operations (see [29]) $S(\mathcal{M})$ is a unital $*$ -algebra over the field \mathbb{C} of complex numbers. It is clear that \mathcal{M} is a $*$ -subalgebra of $S(\mathcal{M})$.

A densely-defined linear operator x affiliated with \mathcal{M} is called *locally measurable* with respect to \mathcal{M} if there is a sequence $\{z_n\}_{n=1}^\infty$ of central projections in \mathcal{M} such that $z_n \uparrow \mathbf{1}$, $z_n(H) \subset \mathfrak{D}(x)$ and $xz_n \in S(\mathcal{M})$ for all $n \in \mathbb{N}$.

The set $LS(\mathcal{M})$ of all locally measurable operators (with respect to \mathcal{M}) is a unital $*$ -algebra over the field \mathbb{C} with respect to the same algebraic operations as in $S(\mathcal{M})$ [31] and $S(\mathcal{M})$ is a $*$ -subalgebra of $LS(\mathcal{M})$. If \mathcal{M} is finite, or if $\dim(\mathcal{Z}(\mathcal{M})) < \infty$, the algebras $S(\mathcal{M})$ and $LS(\mathcal{M})$ coincide [22, Corollary 2.3.5 and Theorem 2.3.16]. If a von Neumann algebra \mathcal{M} is of type *III* and $\dim(\mathcal{Z}(\mathcal{M})) = \infty$, then $S(\mathcal{M}) = \mathcal{M}$ and $LS(\mathcal{M}) \neq \mathcal{M}$ [22, Theorem 2.2.19, Corollary 2.3.15].

For every $x \in S(\mathcal{Z}(\mathcal{M}))$ there exists a sequence $\{z_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $z_n \uparrow \mathbf{1}$ and $xz_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. This means that $x \in LS(\mathcal{M})$. Hence, $S(\mathcal{Z}(\mathcal{M}))$ is a $*$ -subalgebra of $LS(\mathcal{M})$ and $S(\mathcal{Z}(\mathcal{M}))$ coincides with the center of the $*$ -algebra $LS(\mathcal{M})$.

For every subset $E \subset LS(\mathcal{M})$, the sets of all self-adjoint (resp., positive) operators in E will be denoted by E_h (resp. E_+). The partial order in $LS(\mathcal{M})$ is defined by its cone $LS_+(\mathcal{M})$ and is denoted by \leq .

Let $\{z_i\}_{i \in I}$ be a family of pairwise orthogonal non-zero central projections from \mathcal{M} with $\sup_{i \in I} z_i = \mathbf{1}$, where I is an arbitrary set of indexes (in this case, the family $\{z_i\}_{i \in I}$ is called a central decomposition of the unity $\mathbf{1}$). Consider the $*$ -algebra $\prod_{i \in I} LS(z_i \mathcal{M})$ with the coordinate-wise operations and involution and for every $x \in LS(\mathcal{M})$

set

$$\phi(x) := \{z_i x\}_{i \in I}.$$

In [26] it is proven that the mapping ϕ is a $*$ -isomorphism from the $*$ -algebra $LS(\mathcal{M})$ onto $\prod_{i \in I} LS(z_i \mathcal{M})$. From here immediately follows

Proposition 2.1. *Given any central decomposition $\{z_i\}_{i \in I}$ of the unity and any family of elements $\{x_i\}_{i \in I}$ in $LS(\mathcal{M})$ there exists a unique element $x \in LS(\mathcal{M})$ such that $z_i x = z_i x_i$ for all $i \in I$.*

Let x be a closed operator with the dense domain $\mathfrak{D}(x)$ in H , let $x = u|x|$ be the polar decomposition of the operator x , where $|x| = (x^*x)^{\frac{1}{2}}$ and u is a partial isometry in $B(H)$ such that u^*u is the right support $r(x)$ of x . It is known that $x \in LS(\mathcal{M})$ (respectively, $x \in S(\mathcal{M})$) if and only if $|x| \in LS(\mathcal{M})$ (respectively, $|x| \in S(\mathcal{M})$) and $u \in \mathcal{M}$ [22, §§ 2.2, 2.3]. If x is a self-adjoint operator affiliated with \mathcal{M} , then the spectral family of projections $\{E_\lambda(x)\}_{\lambda \in \mathbb{R}}$ for x belongs to \mathcal{M} [22, § 2.1]. A locally measurable operator x is measurable if and only if $E_\lambda^\perp(|x|) \in \mathcal{P}_{fin}(\mathcal{M})$ for some $\lambda > 0$ [22, § 2.2].

Recall that two projections $e, f \in \mathcal{P}(\mathcal{M})$ are called equivalent (notation: $e \sim f$) if there exists a partial isometry $u \in \mathcal{M}$ such that $u^*u = e$ and $uu^* = f$. For every operator $x \in LS(\mathcal{M})$ the left support $l(x)$ and the right support $r(x)$ are always equivalent [22, Prop.2.1.7(iii)], in addition $r(|x|) = r(x) = l(x^*)$. For projections $e, f \in \mathcal{P}(\mathcal{M})$ notation $e \preceq f$ means that there exists a projection $q \in \mathcal{P}(\mathcal{M})$ such that $e \sim q \leq f$.

Now, let us recall the definition of the local measure topology. Firstly, let \mathcal{M} be a commutative von Neumann algebra. Then \mathcal{M} is $*$ -isomorphic to the $*$ -algebra $L^\infty(\Omega, \Sigma, \mu)$ of all essentially bounded measurable complex-valued functions defined on a measure space (Ω, Σ, μ) with the measure μ satisfying the direct sum property (we identify functions that are equal almost everywhere) (see e.g. [30, Ch. III, §1]). The direct sum property of a measure μ means that the Boolean algebra of all projections of the $*$ -algebra $L^\infty(\Omega, \Sigma, \mu)$ is order complete, and for any non-zero $p \in \mathcal{P}(\mathcal{M})$ there exists a non-zero projection $q \leq p$ such that $\mu(q) < \infty$. The direct sum property of a measure μ is equivalent to the fact that the functional $\tau(f) := \int_\Omega f d\mu$ is a semi-finite normal faithful trace on the algebra $L^\infty(\Omega, \Sigma, \mu)$.

Consider the $*$ -algebra $LS(\mathcal{M}) = S(\mathcal{M}) = L^0(\Omega, \Sigma, \mu)$ of all measurable almost everywhere finite complex-valued functions defined on (Ω, Σ, μ) (functions that are equal almost everywhere are identified). Define on $L^0(\Omega, \Sigma, \mu)$ the local measure topology $t(L^\infty(\Omega))$, that is, the Hausdorff vector topology, whose base of neighbourhoods of zero is given by

$$W(B, \varepsilon, \delta) := \{f \in L^0(\Omega, \Sigma, \mu) : \text{there exists a set } E \in \Sigma \text{ such that} \\ E \subseteq B, \mu(B \setminus E) \leq \delta, f\chi_E \in L^\infty(\Omega, \Sigma, \mu), \|f\chi_E\|_{L^\infty(\Omega, \Sigma, \mu)} \leq \varepsilon\},$$

where $\varepsilon, \delta > 0$, $B \in \Sigma$, $\mu(B) < \infty$, $\chi(\omega) = 1$, $\omega \in E$ and $\chi(\omega) = 0$, when $\omega \notin E$.

Convergence of a net $\{f_\alpha\}$ to f in the topology $t(L^\infty(\Omega))$, denoted by $f_\alpha \xrightarrow{t(L^\infty(\Omega))} f$, means that $f_\alpha \chi_B \rightarrow f \chi_B$ in measure μ for every $B \in \Sigma$ with $\mu(B) < \infty$. Note, that the topology $t(L^\infty(\Omega))$ does not change if the measure μ is replaced with an equivalent measure [31].

Now let \mathcal{M} be an arbitrary von Neumann algebra and let φ be a $*$ -isomorphism from $\mathcal{Z}(\mathcal{M})$ onto the $*$ -algebra $L^\infty(\Omega, \Sigma, \mu)$, where μ is a measure satisfying the direct sum property. Denote by $L^+(\Omega, \Sigma, \mu)$ the set of all measurable real-valued functions defined on (Ω, Σ, μ) and taking values in the extended half-line $[0, \infty]$ (functions that are equal almost everywhere are identified). It was shown in [29] that there exists a mapping

$$\mathcal{D}: \mathcal{P}(\mathcal{M}) \rightarrow L^+(\Omega, \Sigma, \mu)$$

that possesses the following properties:

- (D1) $\mathcal{D}(p) \in L_+^0(\Omega, \Sigma, \mu) \iff p \in \mathcal{P}_{fin}(\mathcal{M})$;
- (D2) $\mathcal{D}(p \vee q) = \mathcal{D}(p) + \mathcal{D}(q)$ if $pq = 0$;
- (D3) $\mathcal{D}(u^*u) = \mathcal{D}(uu^*)$ for any partial isometry $u \in \mathcal{M}$;
- (D4) $\mathcal{D}(zp) = \varphi(z)\mathcal{D}(p)$ for any $z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ and $p \in \mathcal{P}(\mathcal{M})$;
- (D5) if $p_\alpha, p \in \mathcal{P}(\mathcal{M})$, $\alpha \in A$ and $p_\alpha \uparrow p$, then $\mathcal{D}(p) = \sup_{\alpha \in A} \mathcal{D}(p_\alpha)$.

A mapping $\mathcal{D}: \mathcal{P}(\mathcal{M}) \rightarrow L^+(\Omega, \Sigma, \mu)$ that satisfies properties (D1)—(D5) is called a *dimension function* on $\mathcal{P}(\mathcal{M})$.

A dimension function \mathcal{D} also has the following properties [29]:

- (D6) if $p_n \in \mathcal{P}(\mathcal{M})$, $n \in \mathbb{N}$, then $\mathcal{D}(\sup_{n \geq 1} p_n) \leq \sum_{n=1}^\infty \mathcal{D}(p_n)$, in addition, when $p_n p_m = 0$, $n \neq m$, the equality holds;
- (D7) if $p_n \in \mathcal{P}_{fin}(\mathcal{M})$, $n \in \mathbb{N}$, $p_n \downarrow 0$, then $\mathcal{D}(p_n) \rightarrow 0$ almost everywhere.

For arbitrary scalars $\varepsilon, \delta > 0$ and a set $B \in \Sigma$, $\mu(B) < \infty$, we set

$$(1) \quad \begin{aligned} V(B, \varepsilon, \delta) &:= \{x \in LS(\mathcal{M}) : \text{there exist } p \in \mathcal{P}(\mathcal{M}), \\ &z \in \mathcal{P}(\mathcal{Z}(\mathcal{M})), \text{ such that } xp \in \mathcal{M}, \|xp\|_{\mathcal{M}} \leq \varepsilon, \\ &\varphi(z^\perp) \in W(B, \varepsilon, \delta), \mathcal{D}(zp^\perp) \leq \varepsilon \varphi(z)\}, \end{aligned}$$

where $\|\cdot\|_{\mathcal{M}}$ is the C^* -norm on \mathcal{M} .

It was shown in [31] that the system of sets

$$\{x + V(B, \varepsilon, \delta) : x \in LS(\mathcal{M}), \varepsilon, \delta > 0, B \in \Sigma, \mu(B) < \infty\}$$

defines a Hausdorff vector topology $t(\mathcal{M})$ on $LS(\mathcal{M})$ such that the sets $\{x + V(B, \varepsilon, \delta)\}$, $\varepsilon, \delta > 0$, $B \in \Sigma$, $\mu(B) < \infty$ form a neighbourhood base of an operator $x \in LS(\mathcal{M})$. It is known that $(LS(\mathcal{M}), t(\mathcal{M}))$ is a complete topological $*$ -algebra, and the topology $t(\mathcal{M})$ does not depend on a choice of dimension function \mathcal{D} and on the choice of $*$ -isomorphism φ (see e.g. [22, §3.5], [31]).

The topology $t(\mathcal{M})$ on $LS(\mathcal{M})$ is called the *local measure topology* (or the *topology of convergence locally in measure*). Note, that in case when $\mathcal{M} = B(H)$ the equality $LS(\mathcal{M}) = \mathcal{M}$ holds [22, §2.3] and the topology $t(\mathcal{M})$ coincides with the uniform topology, generated by the C^* -norm $\|\cdot\|_{B(H)}$.

We will need the following criterion for convergence of nets with respect to this topology.

Proposition 2.2. [22, §3.5] (i) A net $\{p_\alpha\}_{\alpha \in A} \subset \mathcal{P}(\mathcal{M})$ converges to zero with respect to the topology $t(\mathcal{M})$ if and only if there is a net $\{z_\alpha\}_{\alpha \in A} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $z_\alpha p_\alpha \in \mathcal{P}_{fin}(\mathcal{M})$ for all $\alpha \in A$, $\varphi(z_\alpha^\perp) \xrightarrow{t(L^\infty(\Omega))} 0$, and $\mathcal{D}(z_\alpha p_\alpha) \xrightarrow{t(L^\infty(\Omega))} 0$, where $t(L^\infty(\Omega))$ is the local measure topology on $L^0(\Omega, \Sigma, \mu)$ and φ is a $*$ -isomorphism of $\mathcal{Z}(\mathcal{M})$ onto $L^\infty(\Omega, \Sigma, \mu)$.

(ii) A net $\{x_\alpha\}_{\alpha \in A} \subset LS(\mathcal{M})$ converges to zero with respect to the topology $t(\mathcal{M})$ if and only if $E_\lambda^\perp(|x_\alpha|) \xrightarrow{t(\mathcal{M})} 0$ for every $\lambda > 0$, where $E_\lambda^\perp(|x_\alpha|)$ is a spectral projection family for the operator $|x_\alpha|$.

Since the involution is continuous in the topology $t(\mathcal{M})$, the set $LS_h(\mathcal{M})$ is closed in $(LS(\mathcal{M}), t(\mathcal{M}))$. The cone $LS_+(\mathcal{M})$ of positive elements is also closed in $(LS(\mathcal{M}), t(\mathcal{M}))$ [31].

Using Proposition 2.2 it is established the following

Proposition 2.3. [6, Prop.2.3] If $x_\alpha \in LS(\mathcal{M})$, $0 \neq z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, then

$$zx_\alpha \xrightarrow{t(\mathcal{M})} 0 \iff zx_\alpha \xrightarrow{t(z\mathcal{M})} 0.$$

Moreover from Proposition 2.2 immediately follows

Corollary 2.4. If $\{z_\alpha\}_{\alpha \in A} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ and $z_\alpha \downarrow 0$ then $z_\alpha \xrightarrow{t(\mathcal{M})} 0$.

Let us mention the following important property of the topology $t(\mathcal{M})$.

Proposition 2.5. The von Neumann algebra \mathcal{M} is everywhere dense in $(LS(\mathcal{M}), t(\mathcal{M}))$.

Proof. If $x \in LS(\mathcal{M})$, then there exists a sequence $\{z_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $z_n \uparrow \mathbf{1}$ and $xz_n \in S(\mathcal{M})$ for all $n \in \mathbb{N}$. By Corollary 2.4, $z_n \xrightarrow{t(\mathcal{M})} \mathbf{1}$, and therefore $xz_n \xrightarrow{t(\mathcal{M})} x$. Consequently, the algebra $S(\mathcal{M})$ is everywhere dense in $(LS(\mathcal{M}), t(\mathcal{M}))$.

Now let $x \in S(\mathcal{M})$. Then there exists a sequence $\{p_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n^\perp \in \mathcal{P}_{fin}(\mathcal{M})$ and $xp_n \in \mathcal{M}$ for any $n \in \mathbb{N}$. According to (D7) we have that $\mathcal{D}(p_n^\perp) \xrightarrow{t(L^\infty(\Omega))} 0$, therefore, Proposition 2.2(i) implies the convergence $p_n \xrightarrow{t(\mathcal{M})} \mathbf{1}$ (we set $z_n = \mathbf{1}$). Then $xp_n \xrightarrow{t(\mathcal{M})} x$. It means that the algebra \mathcal{M} is everywhere dense in the algebra $S(\mathcal{M})$ with respect to the topology $t(\mathcal{M})$. Thus, the von Neumann algebra \mathcal{M} is everywhere dense in $(LS(\mathcal{M}), t(\mathcal{M}))$. \square

The lattice $\mathcal{P}(\mathcal{M})$ is said to have a countable type if every family of non-zero pairwise orthogonal projections in $\mathcal{P}(\mathcal{M})$ is, at most, countable. A von Neumann algebra is said to be σ -finite if the lattice $\mathcal{P}(\mathcal{M})$ has a countable type. It is shown in [29, Lemma 1.1] that a finite von Neumann algebra \mathcal{M} is σ -finite, provided that the lattice $\mathcal{P}(\mathcal{Z}(\mathcal{M}))$ of central projections has a countable type.

If \mathcal{M} is a commutative von Neumann algebra and $\mathcal{P}(\mathcal{M})$ has a countable type, then \mathcal{M} is $*$ -isomorphic to the $*$ -algebra $L^\infty(\Omega, \Sigma, \mu)$ with $\mu(\Omega) < \infty$. In this case, the topology $t(L^\infty(\Omega))$ is metrizable and has a base of neighbourhoods of zero consisting of the sets $W(\Omega, 1/n, 1/n)$, $n \in \mathbb{N}$. In addition, $f_n \xrightarrow{t(L^\infty(\Omega))} 0 \Leftrightarrow f_n \rightarrow 0$ in measure μ , where $f_n, f \in L^0(\Omega, \Sigma, \mu) = LS(\mathcal{M})$.

We need another basis of neighbourhoods of zero in topology $t(\mathcal{M})$ in the case when the algebra $\mathcal{Z}(\mathcal{M})$ is σ -finite. If φ is $*$ -isomorphism from $\mathcal{Z}(\mathcal{M})$ onto $L^\infty(\Omega, \Sigma, \mu)$, $\mu(\Omega) < \infty$, then $\tau(x) = \int_\Omega \varphi(x) d\mu$ is a faithful normal finite trace on $\mathcal{Z}(\mathcal{M})$. For arbitrary positive scalars $\varepsilon, \beta, \gamma$ set

$$(2) \quad \begin{aligned} V(\varepsilon, \beta, \gamma) &:= \{x \in LS(\mathcal{M}) : \text{there exist } p \in \mathcal{P}(\mathcal{M}), \\ z \in \mathcal{P}(\mathcal{Z}(\mathcal{M})), \text{ such that } xp \in \mathcal{M}, \|xp\|_{\mathcal{M}} \leq \varepsilon, \\ \tau(z^\perp) \leq \beta, \mathcal{D}(zp^\perp) \leq \gamma\varphi(z)\}. \end{aligned}$$

Proposition 2.6. *If the centre $\mathcal{Z}(\mathcal{M})$ of von Neumann algebra \mathcal{M} is σ -finite algebra, then the system of sets given by (2) forms a basis of neighbourhoods of zero in the topology $t(\mathcal{M})$.*

Proof. Let $V(\Omega, \varepsilon, \delta)$ be a neighbourhood of zero of the form (1). If $x \in V(\varepsilon, \delta, \varepsilon)$, then there exist $p \in \mathcal{P}(\mathcal{M})$, $z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, such that $xp \in \mathcal{M}$, $\|xp\|_{\mathcal{M}} \leq \varepsilon$, $\int_\Omega \varphi(z^\perp) d\mu \leq \delta$ and $\mathcal{D}(zp^\perp) \leq \varepsilon\varphi(z)$. The inequality $\int_\Omega \varphi(z^\perp) d\mu \leq \delta$ means that $\varphi(z^\perp) \in W(\Omega, \varepsilon, \delta)$. Hence $x \in V(\Omega, \varepsilon, \delta)$, that implies the inclusion $V(\varepsilon, \delta, \varepsilon) \subset V(\Omega, \varepsilon, \delta)$.

If $x \in V(\Omega, \varepsilon, \delta)$, then there exist $p \in \mathcal{P}(\mathcal{M})$, $z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, such that $\|xp\|_{\mathcal{M}} \leq \varepsilon$, $\varphi(z^\perp) \in W(\Omega, \varepsilon, \delta)$ and $\mathcal{D}(zp^\perp) \leq \varepsilon\varphi(z)$. Inclusion $\varphi(z^\perp) \in W(\Omega, \varepsilon, \delta)$ means that there exists $E \in \Sigma$, such that $\mu(\Omega \setminus E) \leq \delta$ and $0 \leq \varphi(z^\perp)\chi_E \leq \varepsilon$. If $0 < \varepsilon < 1$, then $\varphi(z^\perp)\chi_E = 0$, i.e. $\varphi(z^\perp) \leq \chi_{\Omega \setminus E}$, and therefore $\tau(z^\perp) \leq \delta$, that implies $x \in V(\varepsilon, \delta, \varepsilon)$. \square

3. THE SELF-ADJOINT DERIVATIONS ON ALGEBRA $LS(\mathcal{M})$

Let \mathcal{M} be an arbitrary von Neumann algebra, let \mathcal{A} be a subalgebra in $LS(\mathcal{M})$. A linear mapping $\delta : \mathcal{A} \rightarrow LS(\mathcal{M})$ is called a *derivation* on \mathcal{A} with values in $LS(\mathcal{M})$, if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. Each element $a \in \mathcal{A}$ defines a derivation $\delta_a(x) := [a, x] = ax - xa$ on \mathcal{A} with values in \mathcal{A} . Derivations $\delta_a, a \in \mathcal{A}$, are said to be *inner derivations* on \mathcal{A} .

Now, we list a few properties of derivations on \mathcal{A} which we shall need below.

Lemma 3.1. *If $\mathcal{P}(\mathcal{Z}(\mathcal{M})) \subset \mathcal{A}$, δ is a derivation on \mathcal{A} and $z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, then $\delta(z) = 0$ and $\delta(zx) = z\delta(x)$ for all $x \in \mathcal{A}$.*

Proof. We have that $\delta(z) = \delta(z^2) = \delta(z)z + z\delta(z) = 2z\delta(z)$. Hence, $z\delta(z) = z(2z\delta(z)) = 2z\delta(z)$, that is $z\delta(z) = 0$. Therefore, we have $\delta(z) = 0$. In particular, $\delta(zx) = \delta(z)x + z\delta(x) = z\delta(x)$. \square

Lemma 3.1 immediately implies

Corollary 3.2. *If $z \in \mathcal{P}(\mathcal{Z}(\mathcal{M})) \subset \mathcal{A}$, δ is a derivation on \mathcal{A} , then $\delta(z\mathcal{A}) \subset z\mathcal{A}$ and the restriction $\delta^{(z)}$ of the derivation δ to $z\mathcal{A}$ is a derivation on $z\mathcal{A}$, in addition, if δ is $t(\mathcal{M})$ -continuous, then $\delta^{(z)}$ is $t(z\mathcal{M})$ -continuous.*

Proof. By Lemma 3.1, the inclusion $\delta(z\mathcal{A}) \subset z\mathcal{A}$ holds. Moreover, the linear mapping $\delta^{(z)} : z\mathcal{A} \rightarrow z\mathcal{A}$ has the following property

$$\delta^{(z)}((zx)(zy)) = \delta(zx)zy + zx\delta(zy) = \delta^{(z)}(zx)zy + zx\delta^{(z)}(zy)$$

for all $x, y \in \mathcal{A}$.

If $x_\alpha, x \in z\mathcal{A}$, $x_\alpha \xrightarrow{t(z\mathcal{M})} x$, then $x_\alpha \xrightarrow{t(\mathcal{M})} x$ (Proposition 2.3), and therefore $\delta^{(z)}(x_\alpha) = z\delta(x_\alpha) \xrightarrow{t(\mathcal{M})} z\delta(x) = \delta^{(z)}(x)$, that implies the convergence $\delta^{(z)}(x_\alpha) \xrightarrow{t(z\mathcal{M})} \delta^{(z)}(x)$ (Proposition 2.3). \square

Let \mathcal{A} be a subalgebra in $LS(\mathcal{M})$, $0 \neq e \in \mathcal{P}(\mathcal{M}) \cap \mathcal{A}$, let δ be a derivation on \mathcal{A} and let $\delta^{(e)}$ be a linear mapping from $e\mathcal{A}e$ into $e\mathcal{A}e$ defined by the equality $\delta^{(e)}(x) = e\delta(x)e$, $x \in e\mathcal{A}e \subset \mathcal{A}$. If $e = z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, then $\delta^{(e)}$ coincides with the restriction $\delta^{(z)}$ of the derivation δ to $z\mathcal{A} = z\mathcal{A}z$.

Lemma 3.3. *$\delta^{(e)}$ is a derivation on $e\mathcal{A}e$, in addition, $\delta^{(e)}(e) = 0$.*

Proof. If $x, y \in e\mathcal{A}e$, then $x, y \in \mathcal{A}$ and

$$\begin{aligned} \delta^{(e)}(xy) &= e(\delta(xy))e + e(x\delta(y))e \\ &= (e\delta(x)e)(eye) + (exe)(e\delta(y)e) = \delta^{(e)}(x)y + x\delta^{(e)}(y). \end{aligned}$$

Further, from the equalities

$$\delta^{(e)}(e) = e\delta(e^2)e = e\delta(e)ee + ee\delta(e)e = 2e\delta(e)e = 2\delta^{(e)}(e)$$

it follows that $\delta^{(e)}(e) = 0$. \square

Let \mathcal{A} be a $*$ -subalgebra in $LS(\mathcal{M})$, let δ be a derivation on \mathcal{A} with values in $LS(\mathcal{M})$. Let us define a mapping

$$\delta^* : \mathcal{A} \rightarrow LS(\mathcal{M}),$$

by setting $\delta^*(x) = (\delta(x^*))^*$, $x \in \mathcal{A}$. A direct verification shows that δ^* is also a derivation on \mathcal{A} .

A derivation δ on \mathcal{A} is said to be *self-adjoint*, if $\delta = \delta^*$. Every derivation δ on \mathcal{A} can be represented in the form $\delta = Re(\delta) + iIm(\delta)$, where

$Re(\delta) = (\delta + \delta^*)/2$, $Im(\delta) = (\delta - \delta^*)/2i$ are self-adjoint derivations on \mathcal{A} .

Since $(LS(\mathcal{M}), t(\mathcal{M}))$ is a topological $*$ -algebra, the following result holds.

Lemma 3.4. *If \mathcal{A} is a $*$ -subalgebra in $LS(\mathcal{M})$, then a derivation $\delta : \mathcal{A} \rightarrow LS(\mathcal{M})$ is continuous with respect to the topology $t(\mathcal{M})$ if and only if the self-adjoint derivations $Re(\delta)$ and $Im(\delta)$ are continuous with respect to this topology.*

The following lemma establishes a connection between the property of innerness of derivation δ and the property of innerness of derivations $Re(\delta)$ and $Im(\delta)$.

Lemma 3.5. *Let \mathcal{A} be a $*$ -subalgebra in $LS(\mathcal{M})$. A derivation $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is inner derivation on \mathcal{A} if and only if $Re(\delta)$ and $Im(\delta)$ are inner derivations.*

Proof. If $\delta = \delta_a$, $a \in \mathcal{A}$, $b = Re(a) = (a + a^*)/2$, $c = Im(a) = (a - a^*)/2i$, $x \in \mathcal{A}$, then

$$\begin{aligned} Re(\delta)(x) &= \frac{[a, x] + [a, x^*]^*}{2} = \frac{ax - xa + (ax^* - x^*a)^*}{2} \\ &= icx - ix c = \delta_{ic}(x); \end{aligned}$$

and

$$\begin{aligned} Im(\delta)(x) &= \frac{[a, x] - [a, x^*]^*}{2i} = \frac{ax - xa - (ax^* - x^*a)^*}{2i} \\ &= -ibx + ix b = \delta_{-ib}(x). \end{aligned}$$

Conversely, if $Re(\delta) = \delta_a$, $Im(\delta) = \delta_b$ for some $a, b \in \mathcal{A}$, then for all $x \in \mathcal{A}$ the equality

$$\delta(x) = Re(\delta)(x) + iIm(\delta)(x) = [a, x] + i[b, x] = [a + ib, x] = \delta_{a+ib}(x)$$

holds. \square

Lemma 3.6. *Let \mathcal{A} and δ be the same as in Corollary 3.2 and let $\{z_i\}_{i \in I}$ be a central decomposition of the unity $\mathbf{1}$. If $\delta^{(z_i)} = \delta_{d_i}$, $d_i \in z_i\mathcal{A}$ is an inner derivation on $z_i\mathcal{A}$ for every $i \in I$, then there exists an operator $d \in LS(\mathcal{M})$, such that $\delta(x) = [d, x]$ for all $x \in \mathcal{A}$ and $z_id = d_i$ for every $i \in I$.*

Proof. Since $\{z_i\}_{i \in I}$ is a central decomposition of the unity $\mathbf{1}$, $d_i \in z_i\mathcal{A} \subset z_iLS(\mathcal{M})$, by Proposition 2.1 there exists $d \in LS(\mathcal{M})$, such that $z_id = d_i$ for every $i \in I$. Using Lemma 3.1 and equalities $\delta^{(z_i)}(x) = [d_i, x]$, $x \in z_i\mathcal{A}$, $i \in I$ we have that for all $y \in \mathcal{A}$, $i \in I$ equalities $z_i\delta(y) = \delta^{(z_i)}(z_iy) = [d_i, z_iy] = [z_id, z_iy] = z_i[d, y]$ hold. Since $\sup_{i \in I} z_i = \mathbf{1}$ it follows that $\delta(y) = [d, y]$. \square

Lemma 3.7. *Let δ be a derivation on a subalgebra \mathcal{A} of $LS(\mathcal{M})$ and $\mathcal{P}(\mathcal{M}) \subset \mathcal{A}$. If $p, q \in \mathcal{P}(\mathcal{M})$ and $p\delta(q)p \geq \lambda p$ for some $\lambda > 0$, then*

$$r(qp)\delta(l(qp))r(qp) \geq \lambda r(qp).$$

Proof. Set $e = l(qp)$ and $f = r(qp)$. It is clear that $eq = e$ and $pf = f$. In addition, $e = r((qp)^*) = r(pq)$ and $f = l((qp)^*) = l(pq)$.

Since

$$ef = (eq)(pf) = e(qp)f = l(qp)qpr(qp) = qp = (qp)f = q(pf) = qf$$

and

$$fe = f(pq)e = pq = f(pq) = (fp)q = fq,$$

we have

$$\begin{aligned} f\delta(e)f &= f(f\delta(e))f = f\delta(fe)f - f(\delta(f)e)f \\ &= f\delta(fq)f - f\delta(f)qf = f\delta(f)qf + f(f\delta(q))f - f\delta(f)qf \\ &= f\delta(q)f = f(p\delta(q)p)f \geq \lambda fpf = \lambda f. \end{aligned}$$

□

For every $x \in LS(\mathcal{M})$ we set $s(x) = l(x) \vee r(x)$, where $l(x)$ and $r(x)$ are left and right supports of x respectively.

Let $\mathcal{M}_{h,1} = \{x \in \mathcal{M}_h : \|x\|_{\mathcal{M}} \leq 1\}$. Fix a positive number λ and a self-adjoint derivation $\delta : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$. The set of pairs $\mathcal{S} = \{(p_j, x_j) \in \mathcal{P}(\mathcal{M}) \times \mathcal{M}_{h,1} : p_j \neq 0, j \in J\}$ is called λ -system (for the derivation δ), if

(i). $(p_j \vee s(x_j))(p_i \vee s(x_i)) = 0$ and $(p_j \vee s(x_j))s(\delta(p_i)) = 0$ for $j \neq i$, $j, i \in J$;

(ii). $s(x_j) \sim p_j$ for all $j \in J$;

(iii). $p_j\delta(x_j)p_j \geq \lambda p_j$ for all $j \in J$.

The projection $\bigvee_{j \in J} (p_j \vee s(x_j) \vee s(\delta(p_j)) \vee s(\delta(p_j \vee s(x_j))))$ is called the *support* of the λ -system \mathcal{S} and is denoted by $s(\mathcal{S})$. If λ -system \mathcal{S} is empty set, then we set $s(\mathcal{S}) = 0$.

A λ -system is said to be *maximal* if it does not contained in any larger λ -system.

Theorem 3.8. *Let $\mathcal{S} = \{(p_j, x_j)\}_{j \in J}$ be a maximal λ -system of a self-adjoint derivation $\delta : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$, $g = s(\mathcal{S})^\perp$ and $\delta^{(g)}(x) = g\delta(x)g$, $x \in gLS(\mathcal{M})g$. Then*

$$(3) \quad \delta^{(g)}(g\mathcal{M}g) \subset g\mathcal{M}g.$$

Proof. Let us first prove that

$$(4) \quad \delta^{(g)}(q) \subset g\mathcal{M}g \text{ and } \|\delta^{(g)}(q)\|_{\mathcal{M}} \leq \lambda$$

for any projection $q \in \mathcal{P}(g\mathcal{M}g)$. Since $\delta^* = \delta$, it follows that $\delta(q) \in LS_h(\mathcal{M})$ and therefore $\delta^{(g)}(q) \in LS_h(g\mathcal{M}g)$. Let $\delta^{(g)}(q) \neq 0$ and let p be the spectral projection for $\delta^{(g)}(q)$ corresponding to the interval $(\lambda, +\infty)$. It is clear that $p \leq s(\delta^{(g)}(q)) \leq g$.

Suppose that $p \neq 0$, then

$$(5) \quad 0 \neq \lambda p \leq p\delta^{(g)}(q)p = p\delta(q)p.$$

Since

$$0 \neq p\delta(q)p = \delta(pq)p - \delta(p)qp = \delta(pq)p - \delta(p)(pq)^*,$$

it follows that $qp = (pq)^* \neq 0$. Consequently,

$$e = l(qp) \neq 0 \text{ and } f = r(qp) \neq 0,$$

in addition, $e \sim f$. Since $g = s(\mathcal{S})^\perp$, from the inequalities $f \leq p \leq g$ and $e \leq q \leq g$ it follows that

$$(f \vee e)(p_j \vee s(x_j)) = 0, \quad (f \vee e)s(\delta(p_j)) = 0$$

and

$$(p_j \vee s(x_j))\delta(f) = \delta((p_j \vee s(x_j))f) - \delta(p_j \vee s(x_j))f = 0$$

for all $j \in J$. Moreover, according to (5) and Lemma 3.7 we have that $f\delta(e)f \geq \lambda f$. Thus, the system $\mathcal{S} \cup \{(f, e)\}$ is a λ -system, that contradicts to maximality of the λ -system \mathcal{S} . Consequently, $p = 0$, which implies the inequality $\delta^{(g)}(q) \leq \lambda \mathbf{1}$. Similarly, for projection $(g - q) \leq g$ we obtain that

$$g(\delta(g - q))g = \delta^{(g)}(g - q) \leq \lambda \mathbf{1}.$$

By Lemma 3.3, $g\delta(g)g = 0$, and therefore $-g\delta(q)g \leq \lambda \mathbf{1}$. Thus,

$$-\lambda \mathbf{1} \leq g\delta(q)g \leq \lambda \mathbf{1},$$

i.e. $\delta^{(g)}(q) \in g\mathcal{M}g$ and $\|\delta^{(g)}(q)\|_{\mathcal{M}} \leq \lambda$.

Now, suppose that the inclusion (3) false, i.e. there exists an element $x \in \mathcal{M}_{h,1} \cap (g\mathcal{M}g)$, such that $\delta^{(g)}(x) \in LS_h(\mathcal{M}) \setminus g\mathcal{M}g$. It means that the spectral projection $r = E_{3\lambda}^\perp(\delta^{(g)}(x))$ (or $r = E_{-3\lambda}(\delta^{(g)}(x))$) for $\delta^{(g)}(x)$ corresponding to the interval $(3\lambda, +\infty)$ (respectively, $(-\infty, -3\lambda)$), is not equal to zero. Replacing, if necessary, x on $-x$, we may assume that $r = E_{3\lambda}^\perp(\delta^{(g)}(x)) \neq 0$. It is clear that $r \leq s(\delta^{(g)}(x)) \leq g$ and

$$(6) \quad 0 < 3\lambda r \leq r\delta^{(g)}(x)r = r\delta(x)r.$$

According to (4) we have that $\|\delta^{(g)}(r)\|_{\mathcal{M}} \leq \lambda$, and therefore inclusion $x \in \mathcal{M}_{h,1} \cap g\mathcal{M}g$ and equality

$$r\delta(r)xr + rx\delta(r)r = rg\delta(r)gx + rxg\delta(r)gr$$

imply that

$$\|r\delta(r)xr + rx\delta(r)r\|_{\mathcal{M}} \leq 2\lambda.$$

Consequently,

$$(7) \quad -2\lambda r \leq r\delta(r)xr + rx\delta(r)r \leq 2\lambda r.$$

Using (6) and (7) for $y = rxr$ we obtain that

$$(8) \quad r\delta(y)r = r\delta(rxr)r = r\delta(r)xr + r\delta(x)r + rx\delta(r)r \geq \lambda r > 0,$$

in particular, $y \neq 0$ and $q = s(y) \neq 0$. Let us show that the collection $\mathcal{S} \cup \{(q, y)\}$ forms a λ -system. Since $q \leq r \leq g$, from (8) it follows that $q\delta(y)q \geq \lambda q$, in addition

$$(q \vee s(y))(p_j \vee s(x_j)) = 0 = q(p_j \vee s(x_j)), \quad q\delta(p_j) = 0$$

and

$$(p_j \vee s(x_j))\delta(q) = \delta((p_j \vee s(x_j))q) - \delta(p_j \vee s(x_j))q = 0$$

for all $j \in J$. It means that the set $\mathcal{S} \cup \{(q, y)\}$ is a λ -system, that contradicts to maximality of the λ -system \mathcal{S} . From obtained contradiction follows the validity of inclusion (3). \square

Lemma 3.9. *If $\{x_j\}_{j \in J} \subset \mathcal{M}_{h,1}$, $x_i x_j = 0$, $i \neq j$, $i, j \in J$, then there exists a unique element $x \in \mathcal{M}_{h,1}$, denoted by $\sum_{j \in J} x_j$, such that $xs(x_j) = x_j$ for all $j \in J$ and $\sup_{j \in J} s(x_j) = s(x)$.*

Proof. Denote by \mathcal{A} the commutative von Neumann subalgebra of \mathcal{M} , containing the family $\{x_j\}_{j \in J}$. Since \mathcal{A}_h is an order complete vector lattice, $\{x_j\}_{j \in J}$ is the family of pairwise disjoint element of \mathcal{A}_h and $|x_j| \leq \mathbf{1} \in \mathcal{A}$ for all $j \in J$, it follows that there exists a unique element x of \mathcal{A}_h such that $|x| \leq \mathbf{1}$, $xs(x_j) = x_j$ and $s(x) = \sup_{j \in J} s(x_j)$.

Let y be another element of $\mathcal{M}_{h,1}$, such that $ys(x_j) = x_j$ for all $j \in J$ and $\sup_{j \in J} s(x_j) = s(y)$. Then $(x - y)s(x_j) = x_j - x_j = 0$ for any $j \in J$. Therefore $s(x) = \sup_{j \in J} s(x_j) \leq (r(x - y))^\perp$ and then $x - y = xs(x) - ys(y) = xs(x) - ys(x) = (x - y)s(x) = 0$. \square

Lemma 3.10. *Let $x \in LS_h(\mathcal{M})$, $p, q \in \mathcal{P}(\mathcal{M})$, $\rho, \lambda \in \mathbb{R}$, $\rho < \lambda$,*

$$(9) \quad p x p \leq \rho p$$

and

$$(10) \quad q x q \geq \lambda q.$$

Then $p \preceq q^\perp$ and $q \preceq p^\perp$.

Proof. Set $r = p \wedge q$. Multiplying both parts on both sides of inequalities (9) and (10) by r , we obtain that

$$\lambda r \leq r x r \leq \rho r,$$

that is possible if $r = 0$ only. Therefore $p = p - p \wedge q \sim p \vee q - q \leq q^\perp$, i.e. $p \preceq q^\perp$. Similarly, $q \preceq p^\perp$. \square

Theorem 3.11. *Let $\mathcal{S} = \{(p_j, x_j)\}_{j \in J}$ be a λ -system for a self-adjoint derivation $\delta : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$, let \mathcal{D} be a dimensional function on $\mathcal{P}(\mathcal{M})$. Then*

$$(11) \quad \mathcal{D}(s(\mathcal{S})) \leq 8\mathcal{D}(E_\rho^\perp(\delta(\sum_{j \in J} x_j))) \text{ for any } \rho < \lambda.$$

Proof. Set $x = \sum_{j \in J} x_j$ (see Lemma 3.9) and $p = \sup_{j \in J} p_j$. Let us show that

$$(12) \quad p\delta(x)p \geq \lambda p.$$

Since $(p_j \vee s(x_j))(p_i \vee s(x_i)) = 0$ and $(p_j \vee s(x_j))s(\delta(p_i)) = 0$ for $i \neq j$, it follows that $x_i p_j = x_i s(x_i) p_j = 0$ and $x_i \delta(p_j) = x_i s(x_i) s(\delta(p_j)) \delta(p_j) = 0$ for $i \neq j$. Therefore

$$\delta(x_i) p_j = \delta(x_i p_j) - x_i \delta(p_j) = 0,$$

that implies equality

$$s(\delta(x_i)) p_j = 0 \text{ for } i \neq j.$$

From here and from the equality $p = \sup_{j \in J} p_j$ it follows that

$$s(\delta(x_i)) p = s(\delta(x_i)) p_i.$$

Thus,

$$(13) \quad \delta(x_i) p = \delta(x_i) s(\delta(x_i)) p = \delta(x_i) p_i.$$

By Lemma 3.9, we have that

$$(14) \quad p_i x = p_i s(x) x = (p_i \sup_{j \in J} s(x_j)) x = p_i s(x_i) x = p_i x_i$$

and

$$(15) \quad x_i p = x_i (s(x_i) \sup_{j \in J} p_j) = x_i (s(x_i) p_i) = x_i p_i.$$

Similarly,

$$(16) \quad \begin{aligned} \delta(p_i) x p &= \delta(p_i) (s(\delta(p_i)) \sup_{j \in J} s(x_j)) x p \\ &= \delta(p_i) s(x_i) x p = \delta(p_i) x_i p = \delta(p_i) x_i p_i. \end{aligned}$$

By (13), (14) and (15), we obtain

$$\begin{aligned} \delta(p_i x) p &= \delta(p_i x_i) p = \delta(p_i) x_i p + p_i \delta(x_i) p \\ &= \delta(p_i) x_i p_i + p_i \delta(x_i) p_i = \delta(p_i x_i) p_i, \end{aligned}$$

that according to (16) implies equalities

$$\begin{aligned} p_i (p \delta(x) p) &= p_i \delta(x) p = \delta(p_i x) p - \delta(p_i) x p \\ &= \delta(p_i x_i) p_i - \delta(p_i) x_i p_i = p_i \delta(x_i) p_i. \end{aligned}$$

Hence,

$$(17) \quad p_i (p \delta(x) p) = p_i \delta(x_i) p_i,$$

in particular, the projection p_i commutes with the operator $p \delta(x) p$. Set $y = p \delta(x) p - \lambda p$ and by $y_- = (|y| - y)/2$ denote the negative part of the operator y . Since $p_i y = y p_i$ (see (17)) and $p_i \delta(x_i) p_i \geq \lambda p_i$ (see the definition of λ -system), it follows that

$$(18) \quad y_- p_i = p_i y_- = (p_i (p \delta(x) p - \lambda p))_- \stackrel{(17)}{=} (p_i \delta(x_i) p_i - \lambda p_i)_- = 0$$

for all $i \in J$. From equalities (18) and $p = \sup_{j \in J} p_j$ according to [22, Prop. 2.4.1(ix)], it follows that

$$(19) \quad (pyp)_- = p(p\delta(x)p - \lambda p)_- p = py_- p = 0.$$

Therefore

$$pyp = (pyp)_+ - (pyp)_- \stackrel{(19)}{=} (pyp)_+ \geq 0,$$

that implies inequality (12).

Fix a real number $\rho < \lambda$ and set $q = E_\rho(\delta(x))$. By Lemma 3.10, we obtain

$$(20) \quad p \preceq q^\perp.$$

For every fixed $j \in J$ we have that

$$\delta(p_j) = \delta(p_j^2) = \delta(p_j)p_j + p_j\delta(p_j) = \delta(p_j)p_j + (\delta(p_j)p_j)^*$$

and therefore

$$s(\delta(p_j)) \leq l(\delta(p_j)p_j) \vee p_j,$$

that implies

$$(21) \quad \mathcal{D}(s(\delta(p_j))) \leq \mathcal{D}(l(\delta(p_j)p_j)) + \mathcal{D}(p_j).$$

Since $l(\delta(p_j)p_j) \sim r(\delta(p_j)p_j) \leq p_j$, according to (21) we have

$$(22) \quad \mathcal{D}(s(\delta(p_j))) \leq 2\mathcal{D}(p_j)$$

for all $j \in J$. Similarly,

$$\mathcal{D}(s(\delta(p_j \vee s(x_j)))) \leq 2\mathcal{D}(p_j \vee s(x_j)),$$

and in view of equivalence $p_j \sim s(x_j)$ (see the definition of λ -system) we obtain

$$(23) \quad \mathcal{D}(s(\delta(p_j \vee s(x_j)))) \leq 4\mathcal{D}(p_j).$$

Denote by A the directed set of all finite subsets of J ordered by inclusion and for every $\alpha \in A$ set

$$e_\alpha = \bigvee_{j \in \alpha} (p_j \vee s(x_j) \vee s(\delta(p_j)) \vee s(\delta(p_j \vee s(x_j)))).$$

From properties (D2), (D3) of the dimensional function \mathcal{D} and from inequalities (20), (22) and (23) we have that

$$\begin{aligned} \mathcal{D}(e_\alpha) &\leq \sum_{j \in \alpha} \mathcal{D}(p_j \vee s(x_j) \vee s(\delta(p_j)) \vee s(\delta(p_j \vee s(x_j)))) \\ &\leq \sum_{j \in \alpha} (\mathcal{D}(p_j) + \mathcal{D}(s(x_j)) + \mathcal{D}(s(\delta(p_j))) + \mathcal{D}(s(\delta(p_j \vee s(x_j))))) \\ &\leq 8 \sum_{j \in \alpha} \mathcal{D}(p_j) = 8\mathcal{D}(\sum_{j \in \alpha} p_j) \leq 8\mathcal{D}(p) \leq 8\mathcal{D}(q^\perp). \end{aligned}$$

Since $e_\alpha \uparrow s(\mathcal{S})$ the last inequality and property (D6) of the dimensional function \mathcal{D} imply that

$$\mathcal{D}(s(\mathcal{S})) = \mathcal{D}(\sup_{\alpha \in A} e_\alpha) = \sup_{\alpha \in A} \mathcal{D}(e_\alpha) \leq 8\mathcal{D}(q^\perp).$$

□

4. AUTOMATIC INNERNESS OF CONTINUOUS DERIVATIONS ON ALGEBRA $LS(\mathcal{M})$

Let \mathcal{M} be an arbitrary von Neumann algebra and let $\delta_a(x) = [a, x]$, $a, x \in LS(\mathcal{M})$ be an inner derivation on $LS(\mathcal{M})$. Since $(LS(\mathcal{M}), t(\mathcal{M}))$ is a topological algebra, every derivation δ_a is continuous with respect to the topology $t(\mathcal{M})$.

The main result of this section is the proof of inverse implication.

Theorem 4.1. *Every derivation on the algebra $LS(\mathcal{M})$ continuous with respect to the topology $t(\mathcal{M})$ is inner derivation.*

Proof. Let δ be an arbitrary derivation on the $*$ -algebra $LS(\mathcal{M})$ and let δ be continuous with respect to the topology $t(\mathcal{M})$. By Lemmas 3.4 and 3.5, we may assume that δ is a self-adjoint derivation.

Choose a central decomposition $\{z_i\}_{i \in I}$ of the unity $\mathbf{1}$, such that every Boolean algebra $z_i \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ has a countable type, $i \in I$. By Corollary 3.2 the restriction $\delta^{(z_i)}$ of the derivation δ to $z_i LS(\mathcal{M}) = LS(z_i \mathcal{M})$ is a $t(z_i \mathcal{M})$ -continuous derivation on $LS(z_i \mathcal{M})$. If every derivation $\delta^{(z_i)}$, $i \in I$ is inner, then, by Lemma 3.6, the derivation δ is inner too. Thus, in the proof of Theorem 4.1 we may assume that the centre $\mathcal{Z}(\mathcal{M})$ of von Neumann algebra \mathcal{M} is σ -finite algebra. In this case, there exist a faithful normal finite trace $\tau(x) = \int \varphi(x) d\mu$ on $\mathcal{Z}(\mathcal{M})$ and the vector topology $t(\mathcal{M})$ has the basis of neighbourhoods of zero consists of the sets $V(\varepsilon, \beta, \gamma)$ given by (2) (see Proposition 2.6). Since the derivation δ is $t(\mathcal{M})$ -continuous, for arbitrary $\varepsilon, \beta, \gamma > 0$ there exist $\varepsilon_1, \beta_1, \gamma_1 > 0$, such that $\delta(V(\varepsilon_1, \beta_1, \gamma_1)) \subset V(\varepsilon, \beta, \gamma)$. It is clear that

$$\mathcal{M}_1 := \{x \in \mathcal{M} : \|x\|_{\mathcal{M}} \leq 1\} \subset V(1, \beta_1, \gamma_1) = \varepsilon_1^{-1} V(\varepsilon_1, \beta_1, \gamma_1),$$

and therefore

$$\delta(\mathcal{M}_1) \subset \varepsilon_1^{-1} V(\varepsilon, \beta, \gamma) = V(\varepsilon/\varepsilon_1, \beta, \gamma).$$

Hence, for $t(\mathcal{M})$ -continuous self-adjoint derivation $\delta : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$ and for arbitrary positive numbers β and γ there exists a number $\Delta(\beta, \gamma)$, such that

$$(24) \quad \delta(\mathcal{M}_1) \subset V(\Delta(\beta, \gamma), \beta, \gamma).$$

Let $\mathcal{D}, \varphi, \tau$ be the same as in the definition of the set $V(\varepsilon, \beta, \gamma)$ from (2). Take an arbitrary $2\Delta(\beta, \gamma)$ -system $\mathcal{S} = \{(p_j, x_j)\}_{j \in J}$ for the derivation δ and show that there exists a central projection $z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, such that

$$(25) \quad \tau(z^\perp) \leq \beta \text{ and } \mathcal{D}(zs(\mathcal{S})) \leq 8\gamma\varphi(z).$$

If \mathcal{S} is empty set, then $s(\mathcal{S}) = 0$ and, in this case, relations (25) hold for $z = \mathbf{1}$. Now, let $\mathcal{S} = \{(p_j, x_j)\}_{j \in J}$ is non empty $2\Delta(\beta, \gamma)$ -system. By Lemma 3.9 there exists $x = \sum_{j \in J} x_j \in \mathcal{M}_{h,1}$. From (24) it follows

that $\delta(x) \in V(\Delta(\beta, \gamma), \beta, \gamma)$ for all $\beta, \gamma > 0$. Therefore there exist projections $z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ and $q \in \mathcal{P}(\mathcal{M})$, such that

$$(26) \quad \begin{aligned} \tau(z^\perp) &\leq \beta, \quad \delta(x)q \in \mathcal{M}, \quad \|\delta(x)q\|_{\mathcal{M}} \leq \Delta(\beta, \gamma) \\ &\text{and } \mathcal{D}(zq^\perp) \leq \gamma\varphi(z). \end{aligned}$$

Since $x = x^*$ and $\delta = \delta^*$, it follows that $\delta(x) = (\delta(x))^*$ and, according to (26), we have

$$(27) \quad -\Delta(\beta, \gamma)q \leq q\delta(x)q \leq \Delta(\beta, \gamma)q.$$

Set $\rho = \frac{3}{2} \cdot \Delta(\beta, \gamma)$. Using inequalities (27) and

$$\rho E_\rho^\perp(\delta(x)) \leq E_\rho^\perp(\delta(x))\delta(x)E_\rho^\perp(\delta(x)),$$

we obtain that $E_\rho^\perp(\delta(x)) \preceq q^\perp$ (Lemma 3.10). Consequently, $zE_\rho^\perp(\delta(x)) \preceq zq^\perp$ and, by (11) and (26), we have that

$$\begin{aligned} \mathcal{D}(zs(\mathcal{S})) &\stackrel{(D4)}{=} \varphi(z)\mathcal{D}(s(\mathcal{S})) \stackrel{(11)}{\leq} 8\varphi(z)\mathcal{D}(E_\rho^\perp(\delta(x))) \\ &\stackrel{(D4)}{=} 8\mathcal{D}(zE_\rho^\perp(\delta(x))) \stackrel{(D2), (D3)}{\leq} 8\mathcal{D}(zq^\perp) \stackrel{(26)}{\leq} 8\gamma\varphi(z), \end{aligned}$$

i.e. (25) holds.

For every $n \in \mathbb{N}$ choose a maximal (possible, empty) $2\Delta(2^{-n}, 2^{-n})$ -system \mathcal{S}_n for the derivation δ . Set $q'_n = s(\mathcal{S}_n)^\perp$. By Theorem 3.8, we have that

$$(28) \quad \delta^{(q'_n)}(q'_n\mathcal{M}q'_n) \subset q'_n\mathcal{M}q'_n$$

for all $n \in \mathbb{N}$. Moreover, in view of (25), there exists a projection $z'_n \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, such that

$$(29) \quad \tau(z_n'^\perp) \leq 2^{-n} \text{ and } \mathcal{D}(z_n'q_n'^\perp) \leq 2^{-n+3}\varphi(z_n').$$

It is clear that the sequences of projections $q_n = \bigwedge_{k=n+1}^\infty q'_k$ and $z_n = \bigwedge_{k=n+1}^\infty z'_k$ are increasing, in addition

$$(30) \quad \tau(z_n^\perp) \leq \tau\left(\sup_{k \geq n+1} z_k'^\perp\right) \leq \sum_{k \geq n+1} \tau(z_k'^\perp) \stackrel{(29)}{\leq} \sum_{k \geq n+1} 2^{-k} = 2^{-n}$$

and

$$\begin{aligned}
(31) \quad \mathcal{D}(z_n q_n^\perp) &= \varphi(z_n) \mathcal{D}(\sup_{k \geq n+1} z_n q_k'^\perp) \\
&\leq \varphi(z_n) \mathcal{D}(\sup_{k \geq n+1} z_k' q_k'^\perp) \\
&\stackrel{(D6)}{\leq} \varphi(z_n) \sum_{k \geq n+1} \mathcal{D}(z_k' q_k'^\perp) \\
&\stackrel{(29)}{\leq} \varphi(z_n) \sum_{k \geq n+1} 2^{-k+3} \varphi(z_k') \\
&= \sum_{k \geq n+1} 2^{-k+3} \varphi(z_n z_k') \\
&= \sum_{k \geq n+1} 2^{-k+3} \varphi(z_n) = 2^{-n+3} \varphi(z_n).
\end{aligned}$$

Consider the derivation $\delta^{(q_n)}$ on $q_n LS(\mathcal{M}) q_n$ and show that

$$\delta^{(q_n)}(q_n \mathcal{M} q_n) \subset q_n \mathcal{M} q_n.$$

If $x \in q_n \mathcal{M} q_n$, then $x \in q_{n+1}' \mathcal{M} q_{n+1}'$ and therefore, by (28),

$$\delta^{(q_n)}(x) = q_n \delta(x) q_n = q_n q_{n+1}' \delta(x) q_{n+1}' q_n =$$

$$q_n \delta^{(q_{n+1}')} (x) q_n \subset q_n q_{n+1}' \mathcal{M} q_{n+1}' q_n = q_n \mathcal{M} q_n.$$

Hence, the restriction $\delta^{(q_n)}|_{q_n \mathcal{M} q_n}$ of the derivation $\delta^{(q_n)}$ to $q_n \mathcal{M} q_n$ is a derivation on the von Neumann algebra $q_n \mathcal{M} q_n$. By Sakai-Kadison Theorem [27, Theorem 4.1.6], there exists an element $c_n \in q_n \mathcal{M} q_n$, such that $\delta^{(q_n)}(x) = [c_n, x]$ for all $x \in q_n \mathcal{M} q_n$.

Now, construct a sequence $\{d_n\}$ of \mathcal{M} , such that

$$\begin{aligned}
(32) \quad &q_n d_m q_n = d_n \text{ for all } n \leq m, \\
&\delta^{(q_n)}(x) = [d_n, x] \text{ for all } x \in q_n \mathcal{M} q_n.
\end{aligned}$$

Set $d_1 = c_1$ and assume that elements d_1, \dots, d_n are already constructed. Since $\delta^{(q_n)}(q_n x q_n) = q_n \delta^{(q_{n+1}')} (q_n x q_n) q_n$, it follows that

$$[d_n, q_n x q_n] = q_n [c_{n+1}, q_n x q_n] q_n = [q_n c_{n+1} q_n, q_n x q_n]$$

for any $x \in \mathcal{M}$. Consequently, the element $d_n - q_n c_{n+1} q_n$ contained in the centre of algebra $q_n \mathcal{M} q_n$. By [14, page 18, corollary] there exists an element z of the centre of algebra $q_{n+1} \mathcal{M} q_{n+1}$, such that $d_n - q_n c_{n+1} q_n = z q_n$. Set $d_{n+1} = c_{n+1} + z$. It is clear that

$$(33) \quad \delta^{(q_{n+1}')}|_{q_{n+1} \mathcal{M} q_{n+1}}(x) = [c_{n+1}, x] = [d_{n+1}, x]$$

for all $x \in q_{n+1} \mathcal{M} q_{n+1}$, in addition,

$$d_{n+1} \in q_{n+1} \mathcal{M} q_{n+1} \text{ and } q_n d_{n+1} q_n = q_n c_{n+1} q_n + z q_n = d_n$$

for every $n \in \mathbb{N}$. Moreover, for $k \in \mathbb{N}$, $k < n+1$ the equalities

$$(34) \quad q_k d_{n+1} q_k = q_k q_n d_{n+1} q_n q_k = q_k d_n q_k = \dots = q_k d_{k+1} q_k = d_k$$

hold.

Thus we constructed the sequence $\{d_n\}$ of elements of \mathcal{M} which has property (32).

By [11, Prop.8], the topology $t(\mathcal{M})$ induces on $q_n LS(\mathcal{M}) q_n = LS(q_n \mathcal{M} q_n)$ the topology $t(q_n \mathcal{M} q_n)$, and therefore derivation $\delta^{(q_n)}$ is continuous on $(LS(q_n \mathcal{M} q_n), t(q_n \mathcal{M} q_n))$. By Proposition 2.5, we have that $\overline{q_n \mathcal{M} q_n}^{t(q_n \mathcal{M} q_n)} = LS(q_n \mathcal{M} q_n)$. Consequently, the equality $\delta^{(q_n)}(x) = [d_n, x]$ holds for all $x \in LS(q_n \mathcal{M} q_n)$.

If $n, m \in \mathbb{N}$, $n < m$, then

$$d_m - d_n \stackrel{(32)}{=} q_m d_m q_m - q_n d_m q_n = (q_m - q_n) d_m q_m + q_n d_m (q_m - q_n).$$

Since

$$r((q_m - q_n) d_m q_m) \sim l((q_m - q_n) d_m q_m) \leq q_n^\perp,$$

it follows that

$$\begin{aligned} \mathcal{D}(z_n r(d_m - d_n)) &\leq \mathcal{D}(z_n r((q_m - q_n) d_m q_m) \vee z_n q_n^\perp) \leq \\ &\leq 2\mathcal{D}(z_n q_n^\perp) \stackrel{(31)}{\leq} 2^{-n+4} \varphi(z_n). \end{aligned}$$

From here, in view of (2) and (30), we obtain

$$d_m - d_n \in V(0, 2^{-n}, 2^{-n+4}) \subset V(1/n, 2^{-n}, 2^{-n+4}).$$

It means that $\{d_n\}$ is a Cauchy sequence in $(LS(\mathcal{M}), t(\mathcal{M}))$, and therefore, since the space $(LS(\mathcal{M}), t(\mathcal{M}))$ is complete there exists $d \in LS(\mathcal{M})$, such that $d_n \xrightarrow{t(\mathcal{M})} d$.

Now, let us show that $\delta(x) = [d, x]$ for all $x \in LS(\mathcal{M})$. By (30) and (31) we have that $q_n^\perp \in V(0, 2^{-n}, 2^{-n+3})$ for all $n \in \mathbb{N}$, and therefore $q_n^\perp \xrightarrow{t(\mathcal{M})} 0$. Consequently, $q_n \xrightarrow{t(\mathcal{M})} \mathbf{1}$ and for every $x \in LS(\mathcal{M})$ we have that $q_n x q_n \xrightarrow{t(\mathcal{M})} x$. We just need to use $t(\mathcal{M})$ -continuity of the derivation δ , which implies the following

$$\begin{aligned} \delta(x) &= t(\mathcal{M}) - \lim_{n \rightarrow \infty} (q_n \delta(q_n x q_n) q_n) = t(\mathcal{M}) - \lim_{n \rightarrow \infty} \delta^{(q_n)}(q_n x q_n) = \\ &= t(\mathcal{M}) - \lim_{n \rightarrow \infty} [d_n, q_n x q_n] = [t(\mathcal{M}) - \lim_{n \rightarrow \infty} d_n, t(\mathcal{M}) - \lim_{n \rightarrow \infty} q_n x q_n] = \\ &= [d, x]. \end{aligned}$$

□

Theorem 4.1 allows to give the full description of all derivations on the algebra $LS(\mathcal{M})$ in case when \mathcal{M} is a properly infinite von Neumann algebra.

Theorem 4.2. *If \mathcal{M} is a properly infinite von Neumann algebra, then every derivation on the $*$ -algebra $LS(\mathcal{M})$ is inner.*

Proof. By [6, Theorem 3.3] for properly infinite von Neumann algebra \mathcal{M} every derivation $\delta : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$ is $t(\mathcal{M})$ -continuous.

Consequently, by Theorem 4.1, there exists $d \in LS(\mathcal{M})$, such that $\delta(x) = [d, x]$ for all $x \in LS(\mathcal{M})$. \square

5. DERIVATIONS ON EW^* -ALGEBRAS

In this section we give applications of Theorems 4.1 and 4.2 to the description of continuous derivations on EW^* -algebras. The class of EW^* -algebras (extended W^* -algebras) was introduced in [15] for the purpose of description of $*$ -algebras of unbounded closed operators, which are "similar" to W^* -algebras by their algebraic and order properties.

Let \mathcal{A} be a set of closed, densely defined operators on the Hilbert space H which is a $*$ -algebra under strong sum, strong product, scalar multiplication and the usual adjoint of operators. The set \mathcal{A} is said to be EW^* -algebra [15] if the following conditions hold:

- (i) $(1 + x^*x)^{-1} \in \mathcal{A}$ for every $x \in \mathcal{A}$;
- (ii) the subalgebra \mathcal{A}_b of bounded operators in \mathcal{A} is a W^* -algebra.

In [12] it is given the meaningful connection between EW^* -algebras \mathcal{A} and solid subalgebras of $LS(\mathcal{A}_b)$. Recall [7], that a $*$ -subalgebra \mathcal{A} of $LS(\mathcal{M})$ is called solid if conditions $x \in LS(\mathcal{M})$, $y \in \mathcal{A}$, $|x| \leq |y|$ imply that $x \in \mathcal{A}$. It is clear that every solid $*$ -subalgebra \mathcal{A} in $LS(\mathcal{M})$ with $\mathcal{M} \subset \mathcal{A}$ is an EW^* -algebra and $\mathcal{A}_b = \mathcal{M}$. At the same time, in [12] it is established that every EW^* -algebra \mathcal{A} with the bounded part $\mathcal{A}_b = \mathcal{M}$ is a solid $*$ -subalgebra in the $*$ -algebra $LS(\mathcal{M})$, i.e. $LS(\mathcal{M})$ is the greatest EW^* -algebra of all EW^* -algebras with the bounded part coinciding with \mathcal{M} .

Since every EW^* -algebra \mathcal{A} with the bounded part \mathcal{A}_b is a solid $*$ -subalgebra in $LS(\mathcal{A}_b)$ and $\mathcal{A}_b \subset \mathcal{A}$, according to [6], it follows that in the case when \mathcal{A}_b is a properly infinite W^* -algebra any derivation $\delta: \mathcal{A} \rightarrow LS(\mathcal{A}_b)$ is continuous with respect to the local measure topology $t(\mathcal{A}_b)$.

Now, let \mathcal{A}_b be arbitrary W^* -algebra and let $\delta: \mathcal{A} \rightarrow \mathcal{A}$ be a $t(\mathcal{A}_b)$ -continuous derivation. Since $\mathcal{A}_b \subset \mathcal{A}$, \mathcal{A}_b is everywhere dense in $(LS(\mathcal{A}_b), t(\mathcal{A}_b))$ (Proposition 2.5) and $(LS(\mathcal{A}_b), t(\mathcal{A}_b))$ is a topological $*$ -algebra, there exists a unique $t(\mathcal{A}_b)$ -continuous derivation $\hat{\delta}: LS(\mathcal{A}_b) \rightarrow LS(\mathcal{A}_b)$ such that $\hat{\delta}(x) = \delta(x)$ for all $x \in \mathcal{A}$. By Theorem 4.1 the derivation $\hat{\delta}$ is inner. In [7, Proposition 5.13] it is proved that, if δ is a derivation on a solid $*$ -subalgebra $\mathcal{A} \supset \mathcal{M}$ and $\delta(x) = [w, x]$ for all $x \in \mathcal{A}$ and some $w \in LS(\mathcal{M})$, then there exists $w_1 \in \mathcal{A}$, such that $\delta(x) = [w_1, x]$ for all $x \in \mathcal{A}$, i.e. the derivation δ is inner on the $*$ -subalgebra \mathcal{A} .

Thus the following theorem holds.

Theorem 5.1. (i) Every $t(\mathcal{A}_b)$ -continuous derivation on a EW^* -algebra \mathcal{A} is inner;

(ii) If the bounded part \mathcal{A}_b in an EW^* -algebra \mathcal{A} is a properly infinite W^* -algebra, then every derivation on \mathcal{A} is inner.

Let \mathcal{M} be a semifinite von Neumann algebra acting on the Hilbert space H , τ be a faithful normal semifinite trace on \mathcal{M} . An operator $x \in S(\mathcal{M})$ with the domain $\mathfrak{D}(x)$ is called τ -measurable if for any $\varepsilon > 0$ there exists a projection $p \in \mathcal{P}(\mathcal{M})$ such that $p(H) \subset \mathfrak{D}(x)$ and $\tau(p^\perp) < \varepsilon$.

The set $S(\mathcal{M}, \tau)$ of all τ -measurable operators is a solid $*$ -subalgebra of $LS(\mathcal{M})$ such that $\mathcal{M} \subset S(\mathcal{M}, \tau) \subset S(\mathcal{M})$. If the trace τ is finite, then $S(\mathcal{M}, \tau) = S(\mathcal{M})$. The algebra $S(\mathcal{M}, \tau)$ is a noncommutative version of the algebra of all measurable complex functions f defined on (Ω, Σ, μ) , for which $\mu(\{|f| > \lambda\}) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Let t_τ be the *measure topology* [24] on $S(\mathcal{M}, \tau)$ whose base of neighborhoods of zero is given by

$$U(\varepsilon, \delta) = \{x \in S(\mathcal{M}, \tau) : \text{there exists a projection } p \in \mathcal{P}(\mathcal{M}), \\ \text{such that } \tau(p^\perp) \leq \delta, xp \in \mathcal{M}, \|xp\|_{\mathcal{M}} \leq \varepsilon\}, \quad \varepsilon > 0, \delta > 0.$$

The pair $(S(\mathcal{M}, \tau), t_\tau)$ is a complete metrizable topological $*$ -algebra. Here, the topology t_τ majorizes the topology $t(\mathcal{M})$ on $S(\mathcal{M}, \tau)$ and, if τ is a finite trace, the topologies t_τ and $t(\mathcal{M})$ coincide [22, §§ 3.4, 3.5]. Denote by $t(\mathcal{M}, \tau)$ the topology on $S(\mathcal{M}, \tau)$ induced by the topology $t(\mathcal{M})$. It is not true in general that, if the topologies t_τ and $t(\mathcal{M}, \tau)$ are the same, then the von Neumann algebra \mathcal{M} is finite. Indeed, if $\mathcal{M} = B(H)$, $\dim(H) = \infty$, $\tau = tr$ is the canonical trace on $B(H)$, then $LS(\mathcal{M}) = S(\mathcal{M}) = S(\mathcal{M}, \tau) = \mathcal{M}$, and the two topologies t_τ and $t(\mathcal{M})$ coincide with the uniform topology on $B(H)$.

At the same time, if \mathcal{M} is a finite von Neumann algebra with a faithful normal semifinite trace τ and $t_\tau = t(\mathcal{M}, \tau)$, then $\tau(I) < \infty$ [11].

In [4] it is proven that every t_τ -continuous derivation on $S(\mathcal{M}, \tau)$ is inner. In addition, in [3] it is established that in the case of properly infinite von Neumann algebra \mathcal{M} every derivation on $S(\mathcal{M}, \tau)$ is t_τ -continuous. Thus, in view of Theorem 5.1 we obtain the following

Corollary 5.2. *Let \mathcal{M} be a semifinite von Neumann algebra, let τ be a faithful normal semifinite trace on \mathcal{M} , let δ be a derivation on $S(\mathcal{M}, \tau)$. Then the following conditions are equivalent:*

- (i). δ is $t(\mathcal{M})$ -continuous;
- (ii). δ is t_τ -continuous;
- (iii). δ is inner.

In addition, if \mathcal{M} is a properly infinite von Neumann algebra then every derivation on $S(\mathcal{M}, \tau)$ is inner.

6. AUTOMATIC INNERNESS OF DERIVATIONS ON BANACH \mathcal{M} -BIMODULE OF LOCALLY MEASURABLE OPERATORS

In this section we give one more application of Theorem 4.1 establishing innerness of every derivation on a Banach \mathcal{M} -bimodule of locally measurable operators.

Let \mathcal{M} be a von Neumann algebra. A linear subspace \mathcal{E} of $LS(\mathcal{M})$, is called a \mathcal{M} -bimodule of locally measurable operators if $uxv \in \mathcal{E}$ whenever $x \in \mathcal{E}$ and $u, v \in \mathcal{M}$. If \mathcal{E} is a \mathcal{M} -bimodule of locally measurable operators, $x \in \mathcal{E}$ and $x = v|x|$ is the polar decomposition of operator x then $|x| = v^*x \in \mathcal{E}$ and $x^* = |x|v^* \in \mathcal{E}$. In addition,

$$(35) \quad \text{if } |a| \leq |b|, \ b \in \mathcal{E}, \ a \in LS(\mathcal{M}) \text{ then } a \in \mathcal{E}.$$

Property (35) of a \mathcal{M} -bimodule of locally measurable operators follows from the following proposition.

Proposition 6.1. *Let \mathcal{M} be a von Neumann algebra acting in a Hilbert space H , $a, b \in LS(\mathcal{M})$, $0 \leq a \leq b$. Then $a^{1/2} = cb^{1/2}$ for some $c \in s(b)\mathcal{M}s(b)$, $\|c\|_{\mathcal{M}} \leq 1$, in particular, $a = cbc^*$. In addition, if $c_1 \in \mathcal{M}$ and $a^{1/2} = c_1b^{1/2}$, then $s(b) \cdot c_1 \cdot s(b) = c$.*

Proof. Let us show firstly that $s(a) \leq s(b)$. Since

$$0 \leq (\mathbf{1} - s(b))a(\mathbf{1} - s(b)) \leq (\mathbf{1} - s(b))b(\mathbf{1} - s(b)) = 0,$$

it follows that $(\mathbf{1} - s(b))a^{1/2} = 0$, that implies the equality $(\mathbf{1} - s(b))a = 0$, i.e. $s(b)a = a = a^* = a^*s(b) = as(b)$. Consequently, $s(a) \leq s(b)$.

Thus, passing if necessary to the reduction $s(b)\mathcal{M}s(b)$ we may assume that $s(b) = \mathbf{1}$.

For every $n \in \mathbb{N}$ denote by p_n the spectral projection for the operator b corresponding to the interval $[1/n, n]$. Since $p_n \uparrow s(b) = \mathbf{1}$ it follows that the linear subspace $H_0 = \bigcup_{n=1}^{\infty} p_n H$ is dense in H and $H_0 \subset \mathfrak{D}(b) \cap \mathfrak{D}(b^{1/2})$. Furthermore, according to the inequalities $0 \leq p_n a p_n \leq p_n b p_n \leq n p_n$ we have that $a^{1/2} p_n \in \mathcal{M}$ and $\|a^{1/2} p_n\|_{\mathcal{M}} \leq \sqrt{n}$ for all $n \in \mathbb{N}$. In particular, $H_0 \subset \mathfrak{D}(a^{1/2})$.

Since $b^{1/2} p_n \leq n^{1/2} p_n$ and $b^{1/2}(p_n H) = p_n b^{1/2}(p_n H) \subset p_n H$ for all $n \in \mathbb{N}$ we have $b^{1/2}(H_0) \subset H_0$. Consequently, it is possible to define a linear mapping $d : b^{1/2}(H_0) \rightarrow H$ by setting $d(b^{1/2}\xi) = a^{1/2}\xi$, $\xi \in H_0$. The definition of the operator d is correct since the equality $b^{1/2}\xi = 0$ and inequality

$$\|a^{1/2}\xi\|_H^2 = (a^{1/2}\xi, a^{1/2}\xi) = (a\xi, \xi) \leq (b\xi, \xi) = \|b^{1/2}\xi\|_H^2$$

imply that $a^{1/2}\xi = 0$.

In addition, for every $\xi \in H_0$ we have

$$\|d(b^{1/2}\xi)\|_H^2 = \|a^{1/2}\xi\|_H^2 \leq \|b^{1/2}\xi\|_H^2,$$

i.e. d is a continuous linear operator on $b^{1/2}(H_0)$ and $\|d\|_{b^{1/2}(H_0) \rightarrow H} \leq 1$.

Since $n^{-1}p_n \leq bp_n \leq np_n$, by Proposition [22, 2.4.2] we have $n^{-1/2}p_n \leq b^{1/2}p_n \leq n^{1/2}p_n$. Therefore the restriction of operator $b^{1/2}$ to $p_n(H_0)$ has inverse bounded operator b_n , in addition $n^{-1/2}p_n \leq b_n p_n \leq n^{1/2}p_n$. Hence, $b^{1/2}(p_n H) = p_n H$, that implies the equality $b^{1/2}(H_0) = H_0$.

Thus, the operator d uniquely extends to the Hilbert space H up to a bounded linear operator c , moreover, $\|c\|_{B(H)} \leq 1$ and $cb^{1/2}\xi = a^{1/2}\xi$ for all $\xi \in H_0$.

If u as a unitary operator from the commutant \mathcal{M}' , then $u(p_n H) = p_n H$ for all $n \in \mathbb{N}$ and therefore $u(H_0) = H_0$. If $\eta \in H_0$, then $\eta = b^{1/2}\xi$ for some $\xi \in H_0$ and

$$\begin{aligned} u^{-1}cu\eta &= u^{-1}cub^{1/2}\xi = u^{-1}cb^{1/2}u\xi \\ &= u^{-1}a^{1/2}u\xi = u^{-1}ua^{1/2}\xi = a^{1/2}\xi = cb^{1/2}\xi = c\eta. \end{aligned}$$

Consequently, $u^{-1}cu = c$, that implies the inclusion $c \in \mathcal{M}$.

Since $p_n cb^{1/2}p_n = p_n a^{1/2}p_n$ for all $n \in \mathbb{N}$ and $p_n \uparrow \mathbf{1}$, by Proposition [22, Proposition 2.4.1 (ix)] we have $cb^{1/2} = a^{1/2}$.

If $c_1 \in \mathcal{M}$ and $c_1 b^{1/2} = a^{1/2}$, then the operators c_1 and c coincide on the everywhere dense subspace H_0 and therefore $c_1 = c$.

If $s(b) \neq \mathbf{1}$, $c_1 \in \mathcal{M}$ and $c_1 b^{1/2} = a^{1/2}$, then, using inequalities

$$a^{1/2}s(b) = s(b)a^{1/2} = a^{1/2}$$

and

$$b^{1/2}s(b) = s(b)b^{1/2} = b^{1/2},$$

we obtain $(s(b)c_1 s(b))b^{1/2} = a^{1/2}$. Uniqueness of the operator c in reduction $s(b)\mathcal{M}s(b)$ implies that $s(b) \cdot c_1 \cdot s(b) = c$. \square

Let \mathcal{E} be a \mathcal{M} -bimodule of locally measurable operators. A linear mapping $\delta : \mathcal{M} \rightarrow \mathcal{E}$ is called *derivation*, if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{M}$. A derivation $\delta : \mathcal{M} \rightarrow \mathcal{E}$ is called *inner*, if there exists an element $d \in \mathcal{E}$, such that $\delta(x) = [d, x] = dx - xd$ for all $x \in \mathcal{M}$.

We need the following

Theorem 6.2. [8, Theorem 1] *Let \mathcal{M} be a von Neumann algebra and $a \in LS_h(\mathcal{M})$. Then there exist a self-adjoint operator c in the centre of the $*$ -algebra $LS(\mathcal{M})$ and a family $\{u_\varepsilon\}_{\varepsilon>0}$ of unitary operators from \mathcal{M} such that*

$$(36) \quad |[a, u_\varepsilon]| \geq (1 - \varepsilon)|a - c|.$$

The following theorem establishes innerness of every derivation $\delta : \mathcal{M} \rightarrow \mathcal{E}$ in case of properly infinite von Neumann algebra \mathcal{M} .

Theorem 6.3. *Let \mathcal{M} be a properly infinite von Neumann algebra and let \mathcal{E} be a \mathcal{M} -bimodule of locally measurable operators. Then any derivation $\delta : \mathcal{M} \rightarrow \mathcal{E}$ is inner.*

Proof. By [6, Theorem 4.8] there exists a derivation $\bar{\delta} : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$, such that $\bar{\delta}(x) = \delta(x)$ for all $x \in \mathcal{M}$. By Theorem 4.2, there exists an element $a \in LS(\mathcal{M})$, such that $\bar{\delta}(x) = [a, x]$ for all $x \in LS(\mathcal{M})$. It is clear that $[a, \mathcal{M}] = \bar{\delta}(\mathcal{M}) = \delta(\mathcal{M}) \subset \mathcal{E}$.

Let $a_1 = Re(a)$, $a_2 = Im(a)$. Since $[a^*, x] = -[a, x^*]^* \in \mathcal{E}$ for any $x \in \mathcal{M}$, it follows that $[a_1, x] = [a + a^*, x]/2 \in \mathcal{E}$ and $[a_2, x] = [a - a^*, x]/2i \in \mathcal{E}$ for all $x \in \mathcal{M}$.

By Theorem 6.2 and by taking $\varepsilon = 1/2$ in (36) we obtain that there exist $c_1, c_2 \in \mathcal{Z}_h(LS(\mathcal{M}))$ and unitary operators $u_1, u_2 \in \mathcal{M}$ such that

$$2|[a_i, u_i]| \geq |a_i - c_i|, \quad i = 1, 2.$$

Since $[a_i, u_i] \in \mathcal{E}$ and \mathcal{E} is \mathcal{M} -bimodule we have that $d_i := a_i - c_i \in \mathcal{E}$, $i = 1, 2$ (see (35)). Therefore $d = d_1 + id_2 \in \mathcal{E}$. Since c_1, c_2 are central projections from $LS(\mathcal{M})$ it follows that $\delta(x) = [a, x] = [d, x]$ for all $x \in \mathcal{M}$. \square

Let \mathcal{M} be a von Neumann algebra. If a \mathcal{M} -bimodule of locally measurable operators \mathcal{E} is equipped with a norm $\|\cdot\|_{\mathcal{E}}$, satisfying

$$(37) \quad \|uxv\|_{\mathcal{E}} \leq \|u\|_{\mathcal{M}} \|v\|_{\mathcal{M}} \|x\|_{\mathcal{E}}, \quad x \in \mathcal{E}, \quad u, v \in \mathcal{M},$$

then \mathcal{E} is called a *normed \mathcal{M} -bimodule of locally measurable operators*. If, in addition, $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a Banach space, then \mathcal{E} is called a *Banach \mathcal{M} -bimodule of locally measurable operators*.

Easy to see that for the norm $\|\cdot\|_{\mathcal{E}}$ on a normed \mathcal{M} -bimodule of locally measurable operators \mathcal{E} the following properties hold:

$$(38) \quad \| |a| \|_{\mathcal{E}} = \|a^*\|_{\mathcal{E}} = \|a\|_{\mathcal{E}} \text{ for any } a \in \mathcal{E};$$

$$(39) \quad \|a\|_{\mathcal{E}} \leq \|b\|_{\mathcal{E}} \text{ for any } a, b \in \mathcal{E}, \quad 0 \leq a \leq b;$$

$$(40) \quad \begin{aligned} &\text{If } q \in \mathcal{E} \cap \mathcal{P}(\mathcal{M}), \quad p \in \mathcal{P}(\mathcal{M}), \quad p \preceq q, \\ &\text{then } p \in \mathcal{E} \text{ and } \|p\|_{\mathcal{E}} \leq \|q\|_{\mathcal{E}}. \end{aligned}$$

Proposition 6.4. *If $\{p_k\}_{k=1}^n \subset \mathcal{P}(\mathcal{M}) \cap \mathcal{E}$ then*

$$(41) \quad \bigvee_{k=1}^n p_n \in \mathcal{E} \text{ and } \left\| \bigvee_{k=1}^n p_n \right\|_{\mathcal{E}} \leq \sum_{k=1}^n \|p_k\|_{\mathcal{E}}.$$

Proof. If $p, q \in \mathcal{P}(\mathcal{M}) \cap \mathcal{E}$, then $p \vee q - p \sim q - p \wedge q \leq q$ and therefore, $p \vee q - p \in \mathcal{E}$ and $\|p \vee q - p\|_{\mathcal{E}} \leq \|q\|_{\mathcal{E}}$ (see (40)). Hence, $p \vee q = (p \vee q - p) + p \in \mathcal{E}$ and $\|p \vee q\|_{\mathcal{E}} - \|p\|_{\mathcal{E}} \leq \|p \vee q - p\|_{\mathcal{E}} \leq \|q\|_{\mathcal{E}}$.

For an arbitrary finite set $\{p_k\}_{k=1}^n \subset \mathcal{P}(\mathcal{M}) \cap \mathcal{E}$ proposition (41) is proved using mathematical induction. \square

In Lemmas 6.6-6.9 given below we assume that on a von Neumann algebra \mathcal{M} there is a faithful normal finite trace τ . In this case, the algebra \mathcal{M} is finite. Moreover, $LS(\mathcal{M}) = S(\mathcal{M}) = S(\mathcal{M}, \tau)$, $t(\mathcal{M}) = t_{\tau}$ and $(LS(\mathcal{M}), t(\mathcal{M}))$ is an F -space.

Later we need the following

Proposition 6.5. [22, Prop.3.5.7(i)] *Let \mathcal{M} be a von Neumann algebra with faithful normal finite trace τ and $\{p_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{M})$. Then*

$$p_n \xrightarrow{t(\mathcal{M})} 0 \Leftrightarrow \tau(p_n) \rightarrow 0.$$

Let \mathcal{E} be a Banach \mathcal{M} -bimodule of locally measurable operators in $LS(\mathcal{M})$.

Lemma 6.6. *If $\{p_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{M}) \cap \mathcal{E}$ and the series $\sum_{n=1}^\infty \|p_n\|_{\mathcal{E}}$ converges, then $p = \bigvee_{n=1}^\infty p_n \in \mathcal{E}$ and $\|p\|_{\mathcal{E}} \leq \sum_{n=1}^\infty \|p_n\|_{\mathcal{E}}$.*

Proof. Set $q_n = \bigvee_{k=1}^n p_k$. According to (41), $q_n \in \mathcal{E}$ and $\|q_n\|_{\mathcal{E}} \leq \sum_{k=1}^n \|p_k\|_{\mathcal{E}}$.

Let $n, m \in \mathbb{N}$, $n < m$. By (40) and (41) we have that

$$\begin{aligned} \|q_m - q_n\|_{\mathcal{E}} &= \|q_n \vee \bigvee_{k=n+1}^m p_k - q_n\|_{\mathcal{E}} \\ &= \left\| \bigvee_{k=n+1}^m p_k - q_n \wedge \bigvee_{k=n+1}^m p_k \right\|_{\mathcal{E}} \leq \left\| \bigvee_{k=n+1}^m p_k \right\|_{\mathcal{E}} \leq \sum_{k=n+1}^m \|p_k\|_{\mathcal{E}}. \end{aligned}$$

Consequently, $\{q_n\}$ is a Cauchy sequence in $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, and therefore there exists $q \in \mathcal{E}$, such that $\|q_n - q\|_{\mathcal{E}} \rightarrow 0$, in addition $\|q\|_{\mathcal{E}} \leq \sum_{n=1}^\infty \|p_n\|_{\mathcal{E}}$.

Since

$$\|qp - q_n\|_{\mathcal{E}} = \|qp - q_np\|_{\mathcal{E}} \leq \|p\|_{\mathcal{M}} \|q - q_n\|_{\mathcal{E}},$$

it follows that $qp = q = q^* = pq$. Hence, $s(p - q) \leq p$. Fix $n_0 \in \mathbb{N}$, then for $n > n_0$, we have

$$\begin{aligned} \|q_{n_0}q - q_{n_0}\|_{\mathcal{E}} &= \|q_{n_0}q - q_{n_0}q_n\|_{\mathcal{E}} \\ &\leq \|q_{n_0}\|_{\mathcal{M}} \|q - q_n\|_{\mathcal{E}} \leq \|q - q_n\|_{\mathcal{E}}. \end{aligned}$$

Passing to the limit for $n \rightarrow \infty$, we obtain $q_{n_0}q = q_{n_0}$. Therefore, $q_n(p - q)q_n = 0$ for all $n \in \mathbb{N}$. The inequality $s(p - q) \leq p$ and convergence $q_n \uparrow p$ by [22, Prop. 2.4.1(ix)] imply that $q = p$. \square

Lemma 6.7. *If $\{a_n\}_{n=1}^\infty \subset \mathcal{E}$ and $\|a_n\|_{\mathcal{E}} \rightarrow 0$, then $a_n \xrightarrow{t(\mathcal{M})} 0$.*

Proof. It is sufficient to show that every convergent to zero in the norm $\|\cdot\|_{\mathcal{E}}$ sequence from \mathcal{E} has a subsequence convergent to zero in the topology $t(\mathcal{M})$.

Firstly, consider a sequence $\{p_n\}_{n=1}^\infty \in \mathcal{P}(\mathcal{M}) \cap \mathcal{E}$, such that $\|p_n\|_{\mathcal{E}} \rightarrow 0$. Choose a subsequence $\{p_{n_k}\}_{k=1}^\infty$ so that $\|p_{n_k}\|_{\mathcal{E}} \leq 2^{-k}$. By Lemma 6.6 for the sequence of projections $q_k = \sup_{l \geq k+1} p_{n_l}$ we have $q_k \in \mathcal{E}$ and $\|q_k\|_{\mathcal{E}} \leq 2^{-k}$. If $q = \inf_{k \geq 1} q_k$, then $q \in \mathcal{E}$ and $\|q\|_{\mathcal{E}} \leq \|q_k\|_{\mathcal{E}} \leq 2^{-k}$ for all $k \in \mathbb{N}$, that implies $q = 0$. Consequently, $q_k \downarrow 0$, and therefore $\tau(q_k) \downarrow 0$.

Since $p_{n_{k+1}} \leq q_k$ for all $k \in \mathbb{N}$ we have $\tau(p_{n_k}) \rightarrow 0$, that by Proposition 6.5 implies the convergence $p_{n_k} \xrightarrow{t(\mathcal{M})} 0$. Thus, every sequence

$\{p_n\}_{n=1}^\infty \in \mathcal{P}(\mathcal{M}) \cap \mathcal{E}$ convergent to zero in the norm $\|\cdot\|_{\mathcal{E}}$ automatic converges to zero in the topology $t(\mathcal{M})$.

Now, let $\{a_n\}_{n=1}^\infty \subset \mathcal{E}$ and $\|a_n\|_{\mathcal{E}} \rightarrow 0$. For every $\lambda > 0$ inequality $\lambda E_\lambda^\perp(|a_n|) \leq |a_n| E_\lambda^\perp(|a_n|) \leq |a_n|$ imply that

$$\|E_\lambda^\perp(|a_n|)\|_{\mathcal{E}} \stackrel{(39)}{\leq} \lambda^{-1} \| |a_n| \|_{\mathcal{E}} \stackrel{(38)}{\leq} \lambda^{-1} \|a_n\|_{\mathcal{E}} \rightarrow 0.$$

According to the proven above we have that $E_\lambda^\perp(|a_n|) \xrightarrow{t(\mathcal{M})} 0$. Finally, by Proposition 2.2 (ii) we obtain $a_n \xrightarrow{t(\mathcal{M})} 0$. \square

Lemma 6.8. *If $\{a_n\}_{n=1}^\infty \subset LS(\mathcal{M})$ and $a_n \xrightarrow{t(\mathcal{M})} 0$, then there exists a sequence $\{a_{n_k}\}_{k=1}^\infty$ such that $a_{n_k} = b_k + c_k$, where $b_k \in \mathcal{M}$, $c_k \in LS(\mathcal{M})$, $k \in \mathbb{N}$, $\|b_k\|_{\mathcal{M}} \rightarrow 0$ and $s(|c_k|) \xrightarrow{t(\mathcal{M})} 0$.*

Proof. Since $(LS(\mathcal{M}), t(\mathcal{M}))$ is an F -space there exists a countable basis $\{U_k\}_{k=1}^\infty$ of neighborhoods of zero of the topology $t(\mathcal{M})$.

By Proposition 2.2 (ii) we have $E_\lambda^\perp(|a_n|) \xrightarrow{t(\mathcal{M})} 0$ for every $\lambda > 0$. Therefore, there exists a sequence a_{n_k} such that $E_{1/k}^\perp(|a_{n_k}|) \in U_k$ for all $k \in \mathbb{N}$. Set $b_k = a_{n_k} E_{1/k}^\perp(|a_{n_k}|)$ and $c_k = a_{n_k} E_{1/k}^\perp(|a_{n_k}|)$. It is clear that $b_k \in \mathcal{M}$ and $\|b_k\|_{\mathcal{M}} \leq 1/k$. Since

$$\begin{aligned} |c_k| &= (c_k^* c_k)^{1/2} = (E_{1/k}^\perp(|a_{n_k}|) |a_{n_k}|^2 E_{1/k}^\perp(|a_{n_k}|))^{1/2} \\ &= E_{1/k}^\perp(|a_{n_k}|) |a_{n_k}| E_{1/k}^\perp(|a_{n_k}|) = |a_{n_k}| E_{1/k}^\perp(|a_{n_k}|), \end{aligned}$$

it follows that

$$s(|c_k|) \leq E_{1/k}^\perp(|a_{n_k}|) \in U_k.$$

Since $\{U_k\}_{k=1}^\infty$ is a basis of neighborhoods of zero of the topology $t(\mathcal{M})$ we have $E_{1/k}^\perp(|a_{n_k}|) \xrightarrow{t(\mathcal{M})} 0$, that implies the convergence $\tau(E_{1/k}^\perp(|a_{n_k}|)) \rightarrow 0$ (Proposition 6.5). From the inequality $\tau(s(|c_k|)) \leq \tau(E_{1/k}^\perp(|a_{n_k}|))$ and Proposition 6.5 we obtain $s(|c_k|) \xrightarrow{t(\mathcal{M})} 0$. \square

Lemma 6.9. *Every derivation $\delta : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$ with $\delta(\mathcal{M}) \subset \mathcal{E}$ is $t(\mathcal{M})$ -continuous.*

Proof. Since $(LS(\mathcal{M}), t(\mathcal{M}))$ is an F -space it is sufficient to show that the graph of the linear operator δ is closed.

Suppose that the graph of the operator δ is not closed. Then there exists a sequence $\{a_n\}_{n=1}^\infty \subset LS(\mathcal{M})$ and $0 \neq b \in LS(\mathcal{M})$ such that $a_n \xrightarrow{t(\mathcal{M})} 0$ and $\delta(a_n) \xrightarrow{t(\mathcal{M})} b$.

According to Lemma 6.8 and passing, if necessary, to a subsequence, we may assume that $a_n = b_n + c_n$, where $b_n \in \mathcal{M}$, $c_n \in LS(\mathcal{M})$, $n \in \mathbb{N}$, $\|b_n\|_{\mathcal{M}} \rightarrow 0$ and $s(|c_n|) \xrightarrow{t(\mathcal{M})} 0$ for $n \rightarrow \infty$.

Since the restriction $\delta|_{\mathcal{M}}$ of the derivation δ to the von Neumann algebra \mathcal{M} is a derivation from \mathcal{M} into the Banach \mathcal{M} -bimodule, by

Ringrose Theorem [25] we have $\|\delta(b_n)\|_{\mathcal{E}} \rightarrow 0$. Lemma 6.7 implies that $\delta(b_n) \xrightarrow{t(\mathcal{M})} 0$.

From the inequalities

$$\delta(c_n) = \delta(c_n s(|c_n|)) = \delta(c_n) s(|c_n|) + c_n \delta(s(|c_n|))$$

we have that

$$s(\delta(c_n)) \leq l(\delta(c_n) s(|c_n|)) \vee r(\delta(c_n) s(|c_n|)) \vee l(c_n \delta(s(|c_n|))) \vee r(c_n \delta(s(|c_n|))).$$

Since

$$l(c_n) \sim r(c_n) = s(|c_n|), \quad l(\delta(c_n) s(|c_n|)) \sim r(\delta(c_n) s(|c_n|)) \leq s(|c_n|), \\ r(c_n \delta(s(|c_n|))) \sim l(c_n \delta(s(|c_n|))) \leq l(c_n) \preceq s(|c_n|),$$

it follows that

$$\tau(s(\delta(c_n))) \leq 4\tau(s(|c_n|)).$$

By Proposition 6.5, $\tau(s(|c_n|)) \rightarrow 0$, and therefore $\tau(s(\delta(c_n))) \rightarrow 0$ and $\tau(s(|\delta(c_n)|)) \rightarrow 0$, that implies the convergence $\tau(E_{\lambda}^{\perp}(|\delta(c_n)|)) \rightarrow 0$ for every $\lambda > 0$. Hence by Propositions 2.2 (ii) and 6.5, we obtain $\delta(c_n) \xrightarrow{t(\mathcal{M})} 0$.

Thus, $\delta(a_n) = \delta(b_n) + \delta(c_n) \xrightarrow{t(\mathcal{M})} 0$, that contradicts to the inequality $b \neq 0$. Consequently, the operator δ has a closed graph, therefore δ is $t(\mathcal{M})$ -continuous. \square

Now, we give the main result of this section.

Theorem 6.10. *Let \mathcal{M} be a von Neumann algebra and let \mathcal{E} be a Banach \mathcal{M} -bimodule of local measurable operators. Then any derivation $\delta : \mathcal{M} \rightarrow \mathcal{E}$ is inner. In addition, there exist $d \in \mathcal{E}$ such that $\delta(x) = [d, x]$ for all $x \in \mathcal{M}$ and $\|d\|_{\mathcal{E}} \leq 2\|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$. If $\delta^* = \delta$ or $\delta^* = -\delta$ then d may be chosen so that $\|d\|_{\mathcal{E}} \leq \|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$.*

Proof. According to [6, Theorem 4.8] there exists a derivation $\bar{\delta} : LS(\mathcal{M}) \rightarrow LS(\mathcal{M})$ such that $\bar{\delta}(x) = \delta(x)$ for all $x \in \mathcal{M}$.

Choose a central decomposition of unit $\{z_{\infty}, z_i\}_{i \in J}$ such that $\mathcal{M}z_{\infty}$ is a properly infinite von Neumann algebra and on every von Neumann algebra $\mathcal{M}z_j$ there exists a faithful normal finite trace. By [6, Theorem 3.3] the derivation $\bar{\delta}^{(z_{\infty})} := \bar{\delta}|_{LS(\mathcal{M}z_{\infty})} : LS(\mathcal{M}z_{\infty}) \rightarrow LS(\mathcal{M}z_{\infty})$ is $t(\mathcal{M}z_{\infty})$ -continuous. Lemma 6.9 implies that every derivation $\bar{\delta}^{(z_j)} := \bar{\delta}|_{LS(\mathcal{M}z_j)} : LS(\mathcal{M}z_j) \rightarrow LS(\mathcal{M}z_j)$ is also $t(\mathcal{M}z_j)$ -continuous for all $j \in J$. In this case, according to [6, Cor.2.8], the derivation $\bar{\delta}$ is $t(\mathcal{M})$ -continuous. By Theorem 4.1 the derivation $\bar{\delta}$ is inner. Repeating the proof of Theorem 6.3 we obtain that there exists an element $d \in \mathcal{E}$ such that $\delta(x) = [d, x]$ for all $x \in \mathcal{M}$.

Now, suppose that $\delta^* = \delta$. In this case, $[d + d^*, x] = [d, x] - [d, x^*]^* = \delta(x) - (\delta(x^*))^* = \delta(x) - \delta^*(x) = 0$ for any $x \in \mathcal{M}$. Consequently, the operator $Re(d) = (d + d^*)/2$ commutes with every elements from

\mathcal{M} , and by Proposition 2.5, $Re(d)$ is a central element in the algebra $LS(\mathcal{M})$. Therefore we may suggest that $\delta(x) = [d, x]$, $x \in \mathcal{M}$, where $d = ia$, $a \in \mathcal{E}_h$. According to Theorem 6.2 there exist $c = c^*$ from the centre of the algebra $LS(\mathcal{M})$ and a family $\{u_\varepsilon\}_{\varepsilon>0}$ of unitary operators from \mathcal{M} such that

$$|[a, u_\varepsilon]| \geq (1 - \varepsilon)|a - c|.$$

For $b = ia - ic$ and $\varepsilon = 1/2$ we have

$$|b| = |a - c| \leq 2|[a, u_{1/2}]| = 2|[-id, u_{1/2}]| = 2|[d, u_{1/2}]| \in \mathcal{E}.$$

Consequently, $b \in \mathcal{E}$ (see (35)), moreover,

$$\delta(x) = [d, x] = [ia, x] = [b, x]$$

for all $x \in \mathcal{M}$. Since

$$(1 - \varepsilon)|b| = (1 - \varepsilon)|a - c| \stackrel{(36)}{\leq} |[a, u_\varepsilon]| = |[d, u_\varepsilon]| = |\delta(u_\varepsilon)|,$$

it follows that

$$(1 - \varepsilon)\|b\|_{\mathcal{E}} \stackrel{(39)}{\leq} \|\delta(u_\varepsilon)\|_{\mathcal{E}} \leq \|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$$

for all $\varepsilon > 0$, that implies the inequality $\|b\|_{\mathcal{E}} \leq \|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$.

If $\delta^* = -\delta$, then taking $Im(d)$ instead of $Re(d)$ and repeating previous proof we obtain that $\delta(x) = [b, x]$, where $b \in \mathcal{E}$ and $\|b\|_{\mathcal{E}} \leq \|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$.

Now, suppose that $\delta \neq \delta^*$ and $\delta \neq -\delta^*$. Equality (38) implies that

$$\begin{aligned} \|\delta^*\|_{\mathcal{M} \rightarrow \mathcal{E}} &= \sup\{\|\delta(x^*)^*\|_{\mathcal{E}} : \|x\|_{\mathcal{M}} \leq 1\} \\ &= \sup\{\|\delta(x)\|_{\mathcal{E}} : \|x\|_{\mathcal{M}} \leq 1\} = \|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}. \end{aligned}$$

Consequently,

$$\|Re(\delta)\|_{\mathcal{M} \rightarrow \mathcal{E}} = 2^{-1}\|\delta + \delta^*\|_{\mathcal{M} \rightarrow \mathcal{E}} \leq \|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}.$$

Similarly, $\|Im(\delta)\|_{\mathcal{M} \rightarrow \mathcal{E}} \leq \|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$. Since $(Re(\delta))^* = Re(\delta)$, $(Im(\delta))^* = Im(\delta)$, there exist $d_1, d_2 \in \mathcal{E}$, such that $Re(\delta)(x) = [d_1, x]$, $Im(\delta)(x) = [d_2, x]$ for all $x \in \mathcal{M}$ and $\|d_i\|_{\mathcal{E}} \leq \|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$, $i = 1, 2$. Taking $d = d_1 + id_2$, we have that $d \in \mathcal{E}$, $\delta(x) = (Re(\delta) + i \cdot Im(\delta))(x) = [d_1, x] + i[d_2, x] = [d, x]$ for all $x \in \mathcal{M}$, in addition $\|d\|_{\mathcal{E}} \leq 2\|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$. \square

Note that in [9] in Theorem 16 it is given a variant of Theorem 6.10 for a Banach \mathcal{M} -bimodule of locally measurable operators with additional condition either one of separability or one of reflexivity. Theorem 6.10 given herein removes all these assumptions from a Banach \mathcal{M} -bimodule of locally measurable operators.

Let us point out one of important class of Banach \mathcal{M} -bimodules of locally measurable operators connected with the theory of noncommutative integration.

Let \mathcal{M} be a semifinite von Neumann algebra and τ be a faithful normal semifinite trace on \mathcal{M} . Let $S(\mathcal{M}, \tau)$ be the $*$ -algebra of all τ -measurable operators affiliated with \mathcal{M} .

For each $x \in S(\mathcal{M}, \tau)$ it is possible to define the generalized singular value function

$$\begin{aligned}\mu_t(x) &= \inf\{\lambda > 0 : \tau(E_\lambda^\perp(|x|)) \leq t\} \\ &= \inf\{\|x(\mathbf{1} - e)\|_{\mathcal{M}} : e \in \mathcal{P}(\mathcal{M}), \tau(e) \leq t\},\end{aligned}$$

which allows to define and study a noncommutative version of rearrangement invariant function spaces. It should be noted that at the present time the theory of noncommutative rearrangement invariant spaces has a significant place in researches of Banach spaces (see e.g. [17], [21]).

Let \mathcal{E} be a linear subspace in $S(\mathcal{M}, \tau)$ equipped with a Banach norm $\|\cdot\|_{\mathcal{E}}$ with the following property:

If $x \in S(\mathcal{M}, \tau)$, $y \in \mathcal{E}$ and $\mu_t(x) \leq \mu_t(y)$ then $x \in \mathcal{E}$ and $\|x\|_{\mathcal{E}} \leq \|y\|_{\mathcal{E}}$.

In this case, the pair $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is called *rearrangement invariant spaces of measurable operators*. Every rearrangement invariant spaces of measurable operators is a Banach \mathcal{M} -bimodule [17], and therefore Theorem 6.10 implies the following

Corollary 6.11. *Let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ be a rearrangement invariant spaces of measurable operators, affiliated with a semifinite von Neumann algebra \mathcal{M} and with a faithful semifinite normal trace τ . Then any derivation $\delta : \mathcal{M} \rightarrow \mathcal{E}$ is continuous and there exists $d \in \mathcal{E}$ such that $\delta(x) = [d, x]$ for all $x \in \mathcal{M}$ and $\|d\|_{\mathcal{E}} \leq 2\|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$.*

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DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF UZBEKISTAN, VUZ-
GORODOK, 100174, TASHKENT, UZBEKISTAN

E-mail address: `ber@ucd.uz`

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF UZBEKISTAN,
VUZGORODOK, 100174, TASHKENT, UZBEKISTAN

E-mail address: `chilin@ucd.uz`

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW SOUTH
WALES, SYDNEY, NSW 2052, AUSTRALIA

E-mail address: `f.sukochev@unsw.edu.au`